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AN
INTRODUCTION
TO
ALGEBRA
UPON THE
INDUCTIVE METHOD OF INSTRUCTION.

BY WARREN COLBURN,
AUTHOR OF INTELLECTUAL ARITHMETIC AND SEQUEL TO DITTO.

STEREOTYPED AT THE BOSTON TYPE AND STEREOTYPE FOUNDRY,
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DISTRICT OF MASSACHUSETTS, *to wit.*
District Clerk's Office.

Be it remembered, That on the twenty-fourth day of June, A. D. 1825, in the forty-ninth year of the Independence of the United States of America, Warren Colburn, of the said district, has deposited in this office the title of a book, the right whereof he claims as author, in the words following, *to wit*:—

"An Introduction to Algebra, upon the Inductive Method of Instruction. By Warren Colburn, Author of First Lessons in Arithmetic, &c."

In conformity to the act of the Congress of the United States, entitled "An act for the encouragement of learning, by securing the copies of maps, charts, and books, to the authors and proprietors of such copies, during the times therein mentioned;" and also to an act, entitled "An act supplementary to an act, entitled An act for the encouragement of learning, by securing the copies of maps, charts, and books, to the authors and proprietors of such copies, during the times therein mentioned, and extending the benefits thereof to the arts of designing, engraving, and etching historical and other prints."

JNO. W. DAVIS,
Clerk of the District of Massachusetts.

In order to study this work to advantage, the learner should solve every question in course, and do it *algebraically*. If he finds a question which he can solve as easily without the aid of algebra as with it, he may be assured, this is what the author expected. If he first solves a question, which involves no difficulty, he will understand perfectly what he is about, and he will thereby be enabled to encounter those which are difficult.

When the learner is directed to turn back and do in a new way, something he has done before, let him not fail to do it, for it will be necessary to his future progress; and it will be much better to trace the new principle in what he has done before, than to have a new example for it.

The author has heard it objected to his arithmetics by some, that *they are too easy*. Perhaps the same objection will be made to this treatise on algebra. But in both cases, if they *are* too easy, it is the fault of the subject, and not of the book. For in the First Lessons, there is no explanation; and in the Sequel there is probably less than in any other books, which explain at all. As easy however as they are, the author believes that whoever undertakes to teach them, will find the intellects of his scholars more exercised in studying them, than in studying the most difficult treatise he can put into their hands. When the learner feels, that the subject is above his capacity, he dares not attempt any thing himself, but trusts implicitly to the author; but when he finds it level with his capacity, he readily engages in it. But here there is something more. The learner is required to perform a part himself. He finds a regular part assigned to *him*, and if the teacher does his duty, the learner must give a great many explanations which he does not find in the book.

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ALGEBRA.

Introduction.

THE operations explained in Arithmetic are sufficient for the solution of all questions in numbers, that ever occur; but it is to be observed, that in every question there are two distinct things to be attended to; first, to discover, by a course of reasoning, what operations are necessary; and, secondly, to perform those operations. The first of these, to a certain extent, is more easily learnt than the second; but, after the method of performing the operations is understood, all the difficulty in solving abstruse and complicated questions consists in discovering how the operations are to be applied.

It is often difficult, and sometimes absolutely impossible to discover, by the ordinary modes of reasoning, how the fundamental operations are to be applied to the solution of questions. It is our purpose, in this treatise, to show how this difficulty may be obviated.

It has been shown in Arithmetic, that ordinary calculations are very much facilitated by a set of arbitrary signs, called *figures;* it will now be shown that the reasoning, previous to calculation, may receive as great assistance from another set of arbitrary signs.

Some of the signs have already been explained in Arithmetic; they will here be briefly recapitulated.

(=) Two horizontal lines are used to express the words "*are equal to,*" or any other similar expression.

(+) A cross, one line being horizontal and the other perpendicular, signifies "*added to.*" It may be read *and*, *more*, *plus*, or any similar expression; thus, $7 + 5 = 12$, is read 7 and 5 are 12, or 5 added to 7 *is equal to* 12, or 7 *plus* 5 is equal to 12. *Plus* is a Latin word signifying *more.*

(—) A horizontal line, signifies *subtracted from.* It is sometimes read *less* or *minus*. *Minus* is Latin, signifying *less*. Thus

$14 - 6 = 8$, is read 6 subtracted from 14, or 14 *less* 6, or 14 *minus* 6 is equal to 8.

Observe that the signs $+$ and $-$ affect the numbers which they stand immediately before, and no others. Thus

$$14 - 6 + 8 = 16; \text{ and } 14 + 8 - 6 = 16;$$

and $8 - 6 + 14 = 16$; and, in fine, $-6 + 8 + 14 = 16$. In all these cases the 6 only is to be subtracted, and it is the same, whether it be first subtracted from one of the numbers, and then the rest be added, or whether all the others be added and that be subtracted at last.

($\times$) (.) An inclined cross, or a point, is used to express multiplication; thus, $5 \times 3 = 15$, or $5 . 3 = 15$.

($\div$) A horizontal line, with a point above and another below it, is used to express division. Thus $15 \div 3 = 5$, is read 15 *divided by* 3 is equal to 5.

But division is more frequently expressed in the form of a fraction (*Arith.* Art. XVI. Part II.), the divisor being made the denominator, and the dividend the numerator. Thus $\frac{15}{3} = 5$, is read 15 divided by 3 is equal to 5, or one third of 15, is 5, or 15 contains 3, 5 times.

Example. $6 \times 9 + 15 - 3 = 7 . 8 - \frac{16}{4} + 14$.

This is read, 9 times 6 and 15 less 3 are equal to 8 times 7 less 16 divided by 4, and 14.

To find the value of each side; 9 times 6 are 54 and 15 are 69, less 3 are 66. Then 8 times 7 are 56, less 16 divided by 4, or 4 are 52, and 14 more are 66.

In questions proposed for solution, it is always required to find one or more quantities which are unknown; these, when found, are the answer to the question. It will be found extremely useful to have signs to express these unknown quantities, because it will enable us to keep the object more steadily and distinctly in view. We shall also be able to represent certain operations upon them by the aid of signs, which will greatly assist us in arriving at the result.

Algebraic signs are in fact nothing else than an abridgment of common language, by which a long process of reasoning is presented at once in a single view.

The signs generally used to express the unknown quantities above mentioned are some of the last letters of the alphabet, as x, y, z, &c.

I. 1. Two men, A and B, trade in company, and gain 267 dollars, of which B has twice as much as A. What is the share of each?

In this example the unknown quantities are the particular shares of A and B.

Let x represent the number of dollars in A's share, then $2x$ will represent the number of dollars in B's share. Now these added together must make the number of dollars in both their shares, that is, 267 dollars.

$$x + 2x = 267$$

Putting all the x's together, $3x = 267$

If $3x$ are 267, $1x$ is $\frac{1}{3}$ of 267 in the same manner as if 3 oxen were worth \$267, 1 ox would be worth $\frac{1}{3}$ of it.

$$x = 89 = \text{A's share.}$$
$$2x = 178 = \text{B's share.}$$

2. Four men, A, B, C, and D, found a purse of money containing \$325, but not agreeing about the division of it, each took as much as he could get; A got a certain sum, B got 5 times as much; C, 7 times as much; and D, as much as B and C both. How many dollars did each get?

Let x represent the number of dollars that A got; then B got $5x$, C $7x$, and D $(5x + 7x) = 12x$. These, added together, must make \$325, the whole number to be divided.

$$x + 5x + 7x + 12x = 325$$

Putting all the x's together, $25x = 325$

$$x = 13 = \text{A's share.}$$
$$5x = 65 = \text{B's ``}$$
$$7x = 91 = \text{C's ``}$$
$$12x = 156 = \text{D's ``}$$

Note. All examples of this kind in algebra admit of proof. In this case the work is proved by adding together the several shares. If they are equal to the whole sum, 325, the work is right. As the answers are not given in this work, it will be well for the learner always to prove his results.

In the same manner perform the following examples.

3. Said A to B, my horse and saddle together are worth \$130, but the horse is worth 9 times as much as the saddle. What is the value of each?

4. Three men, A, B, and C, trade in company, A puts in a certain sum, B puts in 3 times as much, and C puts in as much

as A and B both; they gain \$656. What is each man's share of the gain?

5. A gentleman, meeting 4 poor persons, distributed 60 cents among them, giving the second twice, the third three times, and the fourth four times as much as the first. How many cents did he give to each?

6. A gentleman left 11000 crowns to be divided between his widow, two sons, and three daughters. He intended that the widow should receive twice the share of a son, and that each son should receive twice the share of a daughter. Required the share of each.

Let x represent the share of a daughter, then $2x$ will represent the share of a son, &c.

7. Four gentlemen entered into a speculation, for which they subscribed \$4755, of which B paid 3 times as much as A, and C paid as much as A and B, and D paid as much as B and C. What did each pay?

8. A man bought some oxen, some cows, and some sheep for \$1400; there were an equal number of each sort. For the oxen he gave \$42 apiece, for the cows \$20, and for the sheep \$8 apiece. How many were there of each sort?

In this example the unknown quantity is the number of each sort, but the number of each sort being the same, one character will express it.

Let x denote the number of each sort.

Then x oxen, at \$42 apiece, will come to $42x$ dolls., and x cows, at \$20 apiece, will come to $20x$ dolls., and x sheep, at \$8 apiece, will come to $8x$ dolls. These added together must make the whole price.

$$42x + 20x + 8x = 1400$$

Putting the x's together, . . $70x = 1400$

Dividing by 70, ... $x = 20$

Ans. 20 of each sort.

9. A man sold some calves and some sheep for \$374, the calves at \$5, and the sheep at \$7 apiece; there were three times as many calves as sheep. How many were there of each?

Let x denote the number of sheep; then $3x$ will denote the number of calves.

Then x sheep, at \$7 apiece, will come to $7x$ dolls., and $3x$ calves, at \$5 apiece, will come to 5 times $3x$ dolls., that is, $15x$ dolls.

These added together must make the whole price.

$$7x + 15x = 374$$

Putting the x's together, $22x = 374$

Dividing by 22, $x = 17 =$ number of sheep.

$3x = 51 =$ " calves.

The learner must have remarked by this time, that when a question is proposed, the first thing to be done, is to find, by means of the unknown quantity, an expression which shall be equal to a given quantity, and then from that, by arithmetical operations, to deduce the value of the unknown quantity.

This expression of equality between two quantities, is called an *equation*. In the last example, $7x + 15x = 374$ is an *equation*.

The quantity or quantities on the left of the sign $=$ are called the *first member*, those on the right, the *second member* of the equation. $(7x + 15x)$ is the first member of the above equation, and 374 is the second member.

Quantities connected by the signs $+$ and $-$ are called *terms*. $7x$ and $15x$ are terms in the above equation.

The figure written before a letter showing how many times the letter is to be taken, is called the *coefficient* of that letter. In the quantities $7x$, $15x$, $22x$; 7, 15, 22, are coefficients of x.

The process of forming an equation by the conditions of a question, is called *putting the question into an equation.*

The process by which the value of the unknown quantity is found, after the question is put into an equation, is called *solving or reducing the equation.*

No rules can be given for putting questions into equations; this must be learned by practice; but rules may be found for solving most of the equations that ever occur.

After the preceding questions were put into equation, the first thing was to reduce all the terms containing the unknown quantity to one term, which was done by adding the coefficients. As $7x + 15x$ are $22x$. Then, since $22x = 374$, $1x$ must be equal to $\frac{1}{22}$ of 374. That is,

When the unknown quantity in one member is reduced to one term, and stands equal to a known quantity in the other, its value is *found by dividing the known quantity by the coefficient of the unknown quantity.*

10. A man bought some oranges, some lemons, and some pears, for 156 cents ; the oranges at 6 cents each, the lemons at 4 cents, and the pears at 3 cents ; there was an equal number of each sort. Required the number of each.

11. In fencing the side of a field, the length of which was 450 yards, two workmen were employed ; one fenced 9 yards, and the other 6 yards per day. How many days did they work ?

12. Three men built 780 rods of fence ; the first built 9 rods per day, the second 7, and the third 5 ; the second worked three times as many days as the first, and the third, twice as many days as the second. How many days did each work ?

13. A man bought some oxen, some cows, and some calves for $348 ; the oxen at $38 each, the cows at $18, and the calves at $4. There were three times as many cows as oxen, and twice as many calves as cows. How many were there of each sort ?

14. A merchant bought a quantity of flour for $132 ; for one half of it he gave $5 per barrel, and for the other half $7. How many barrels were there in the whole ?

Let x denote one half the number of barrels.

15. From two towns, which are 187 miles apart, two travellers set out at the same time with an intention of meeting ; one of them travels at the rate of 8, the other of 9 miles each day. In how many days will they meet ?

II. 1. A cask of wine was sold for $45, which was only $\frac{3}{4}$ of what it cost. Required the cost.

Let x denote the cost.

Three fourths of x may be written $\frac{3}{4}x$ or $\frac{3x}{4}$. The latter is preferable.

$$\frac{3x}{4} = 45$$

$$\frac{1x}{4} = 15$$

$$x = 60$$

Ans. $60.

If $\frac{3}{4}$ of x comes to 45, then $\frac{1x}{4}$ must come to $\frac{1}{3}$ of 45, or 15, and x will be 4 times 15, or 60.

A better method.

$$\frac{3x}{4} = 45$$

$$3x = 45 \times 4 = 180$$

$$x = 60$$

Observe, that $\frac{3x}{4}$ is the same as $\frac{1}{4}$ of $3x$. Now if $\frac{1}{4}$ of $3x$ is 45, $3x$ itself must be 4 times 45, or 180; $3x$ being 180, x must be $\frac{1}{3}$ of 180, which is 60.

2. A man, being asked his age, answered, that if its half and its third were added to it the sum would be 88. What was his age?

Let x denote his age; then,

$$x + \frac{x}{2} + \frac{x}{3} = 88$$

Reducing the terms to a common denominator, $\left.\right\}$ $\frac{6x}{6} + \frac{3x}{6} + \frac{2x}{6} = 88$

Adding them together, $\frac{11x}{6} = 88$

$\frac{1}{6}$ of $11x$ being 88, $11x$ will be 6 times 88, $11x = 528$

Dividing by 11, $x = 48$

Ans. 48 years.

3. If $\frac{5}{8}$ of a hogshead of wine cost $65; what will a hogshead cost at that rate?

4. There is a pole $\frac{1}{2}$ and $\frac{1}{3}$ under water, and 5 feet out of water; what is the length of the pole?

Let x denote the whole length. Then $\frac{x}{2} + \frac{x}{3} + 5$ must be equal to the whole length. Hence,

$$x = \frac{x}{2} + \frac{x}{3} + 5$$

Reducing to a common denominator,

$$\frac{6x}{6} = \frac{3x}{6} + \frac{2x}{6} + 5$$

Adding together, $\frac{6x}{6}=\frac{5x}{6}+5$

Since the two members are equal, if $\frac{5x}{6}$ be subtracted from both, they will still be equal; hence,

$$\frac{x}{6}=5$$

$$\text{and } x=30 \qquad \textit{Ans.}\ 30 \text{ feet.}$$

Proof. One half of 30 is 15, and one third of thirty is 10. Now $30=15+10+5$.

There is another mode of reducing the above equation which in most cases is to be preferred. It is the same in principle.

If both members of an equation be multiplied by the same number, they evidently will still be equal.

In the equation,

$$x=\frac{x}{2}+\frac{x}{3}+5.$$

First multiply both members by 2, the denominator of one of the fractions, and it becomes,

$$2x=x+\frac{2x}{3}+10.$$

Next multiply both members by 3, the denominator of the other fraction, and it becomes,

$$6x=3x+2x+30$$
$$\text{or } 6x=5x+30.$$

Subtracting $5x$ from both members,

$$x=30 \text{ as before.}$$

5. In an orchard of fruit trees $\frac{1}{2}$ of them bear apples, $\frac{1}{4}$ of them pears, $\frac{1}{8}$ of them plums, 7 bear peaches, and 3 bear cherries; these are all the trees in the orchard. How many are there?

6. A farmer, being asked how many sheep he had, answered, he had them in four pastures; in the first he had $\frac{1}{3}$ of them, in the second $\frac{1}{4}$, in the third $\frac{1}{6}$, and in the fourth he had 24 sheep. How many had he in the whole?

7. A person having spent $\frac{1}{2}$ and $\frac{1}{3}$ of his money, had $\$26\frac{2}{3}$ left. How much money had he at first?

8. A man driving his geese to market, was met by another, who said good morrow, master, with your hundred geese; said he, I have not a hundred, but if I had as many more, and half

as many more, and two geese and a half, I should have a hundred. How many had he ?

9. A and B having found a bag of money, disputed about the division of it. A said that $\frac{1}{2}$ and $\frac{1}{3}$ and $\frac{1}{4}$ of the money made $130, and if B could tell how much money there was, he should have it all, otherwise none of it. How much money was there in the bag ?

10. Upon measuring the corn produced in a field, being 96 bushels, it appeared that it had yielded only one third part more than was sown. How much was sown ?

11. A man sold 96 loads of hay to two persons ; to the first $\frac{1}{3}$, and to the second $\frac{3}{5}$ of what his stack contained. How many loads did the stack contain at first ?

12. A and B talking of their ages, A says to B if $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$ of my age be added to my age, and 2 years more, the sum will be twice my age. What was his age ?

13. What sum of money is that whose $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{5}$ part added together amount to £9 ?

14. The account of a certain school is as follows : $\frac{1}{16}$ of the boys learn geometry, $\frac{3}{8}$ learn grammar, $\frac{3}{10}$ learn arithmetic, $\frac{3}{20}$ learn spelling, and 9 learn to read. What is the number of scholars in the school ?

15. There is a fish whose head weighs 9 lb. his tail weighs as much as his head and half his body, and his body weighs as much as his head and tail both. What is the weight of the fish ?

Represent the weight of the body by x.

16. There is a fish whose head is 4 inches long, the tail is twice the length of the head, added to $\frac{2}{5}$ of the length of the body, and the body is as long as the head and tail both. What is the whole length of the fish ?

17. A and B talking of their ages, A says to B, your age is twice and three-fifths of my age, and the sum of our ages is 54. What is the age of each ?

18. A man divided $40 between two persons ; to the first he gave a certain sum, and to the second only $\frac{3}{5}$ as much. How much did he give to each ?

Let x denote the share of the first, $\frac{3x}{5}$ will denote the share of the second. These added together must make $40.

$$x + \frac{3x}{5} = 40$$

Multiplying by 5, $5x + 3x = 200$

Adding together, $8x = 200$
Dividing by 8, $x = 25 =$ share of the first.
$\frac{3x}{5} = 15 =$ " second.

19. Three persons are to share $290 in the following manner: the second is to have two thirds, and the third three fourths as much as the first. What is the share of each?

20. A farmer wishes to mix 100 bushels of provender, consisting of rye, barley, and oats, so that it may contain $\frac{5}{7}$ as much barley as oats, and $\frac{1}{2}$ as much rye as barley. How much of each must there be in the mixture?

21. Divide 40 apples between two boys in the proportion of 3 to 2.

The proportion 3 to 2 signifies that the second will have $\frac{2}{3}$ as many as the first.

22. A gentleman gave to 3 persons £98. The second received five-eighths of the sum given to the first, and the third one-fifth of what the second had. What did each receive?

23. A prize of $1280 was divided between two persons, in the proportion of 9 to 7. What was the share of each?

24. Three men trading in company, put in money in the following proportion; the first 3 dollars as often as the second 7, and the third 5. They gain $960. What is each man's share of the gain?

Observe, the second put in $\frac{7}{3}$ of what the first put in, and the third put in $\frac{5}{3}$.

25. Three men traded together; the first put in $700, the second $450, and the third $950. They gained $420. What was the share of each?

Observe, the second put in $\frac{450}{700} = \frac{45}{70} = \frac{9}{14}$ of what the first put in, &c.

III. 1. Two men, A and B, hired a pasture together for \$55, and A was to pay \$13 more than B. What did each pay?

Suppose B paid x dollars; A was to pay 13 dollars more; therefore he paid $x + 13$. These put together must make the whole 55 dollars.

$$x + x + 13 = 55$$

Putting the x's together,

$$2x + 13 = 55$$

It appears that $2x$ is not so much as 55 by 13, therefore taking 13 from 55,

$$2x = 55 - 13$$

$$2x = 42$$

Dividing by 2, $x = 21 =$ B's share.

B's share is \$21, and A's, being 13 more, is \$34,

$$x + 13 = 21 + 13 = 34 = \text{A's share.}$$

Proof. $34 + 21 = 55$ the whole sum.

2. A man bought a horse and chaise for \$300; the horse cost \$28 more than the chaise. What was the price of each?

3. A man bequeathed his estate of \$12000 to his son and daughter; the son was to have \$2350 more than the daughter. What was the share of each?

4. A father who has three sons, leaves them 16000 crowns. The will specifies that the eldest shall have 2000 crowns more than the second, and that the second shall have 1000 more than the youngest. What is the share of each?

Let x denote the number of crowns in the share of the youngest, then $x + 1000$ will denote the share of the second, and $x + 1000 + 2000$ will denote the share of the eldest. These added together must make the whole sum.

$$x + x + 1000 + x + 1000 + 2000 = 16000$$

Putting together the x's and the numbers,

$$3x + 4000 = 16000$$

It appears that $3x$ is not so much as 16000 by 4000, therefore subtracting 4000 from 16000,

$$3x = 16000 - 4000$$

$$3x = 12000$$

Dividing by 3, $x = 4000 =$ share of the youngest.

The share of the youngest is 4000 crowns; add to this 1000, it makes 5000, the share of the second,

$$x + 1000 = 5000 = \text{share of the second.}$$

Add 2000 more, it makes 7000, the share of the eldest,

$$x + 1000 + 2000 = 7000 = \text{share of the eldest.}$$

Proof. The several shares added make 16000 crowns which is the whole estate.

5. A draper bought three pieces of cloth, which together measured 159 yards; the second piece was 15 yards longer than the first, and the third was 24 yards longer than the second. What was the length of each?

6. A gentleman bequeathed an estate of \$65000 to his wife, two sons, and three daughters. The wife was to have \$2000 less than the elder son, and \$3000 more than the younger son; and the portion of each of the daughters was \$3500 less than that of the younger son. Required the share of each.

The 1st example may be performed differently. Let x denote the number of dollars paid by A; B paid \$13 less, therefore $x - 13$ will represent the number of dollars paid by B. These added together must make the whole.

$$x + x - 13 = 55$$

Putting the x's together, $2x - 13 = 55$

It appears that $2x$ is more than 55 by 13, therefore add 13 to 55 to make $2x$,

$$2x = 55 + 13$$
$$2x = 68$$

Dividing by 2, $x = 34 = \text{A's share.}$

This gives A's share \$34, from which subtract \$13, and it gives B's share \$21, as before,

$$x - 13 = 21 = \text{B's share.}$$

In the same manner perform the 2d and 3d. The 4th may be solved in a similar manner.

Let the elder son's share be represented by x. The second son's share, being \$2000 less, will be $x - 2000$. The younger son's share, being \$1000 less still, will be $x - 2000 - 1000$. These added together must make the whole sum.

$$x + x - 2000 + x - 2000 - 1000 = 16000$$

Putting the x's together and the numbers together,

$$3x - 5000 = 16000.$$

It appears that $3x$ is more than 16000 by 5000, therefore add 5000 to 16000,

$$3x = 16000 + 5000$$
$$3x = 21000$$

Dividing by 3, $x = 7000$

The elder son's share is \$7000, as before. The others may be easily found from this.

Again, let x denote the second son's share. The elder son's, being \$2000 more, will be $x + 2000$. The younger son's, being \$1000 less, will be $x - 1000$. These added together must make the whole.

$$x + 2000 + x + x - 1000 = 16000$$

Putting the x's together and the numbers together,

$$3x + 1000 = 16000$$
$$3x = 16000 - 1000$$
$$3x = 15000$$
$$x = 5000.$$

The second son's share is \$5000, as before. From this the rest are easily found.

Perform the 5th and 6th in a similar way.

7. At a certain election 943 men voted, and the candidate chosen had a majority of 65. How many voted for each?

8. A person employed 4 workmen; to the first of whom he gave 2 shillings more than to the second; to the second 3 shillings more than to the third; and to the third 4 more than to the fourth. Their wages amounted to 32 shillings. What did each receive?

9. A cask, which held 146 gallons, was filled with a mixture of brandy, wine, and water. In it there were 15 gallons of wine more than there were of brandy, and as much water as both wine and brandy. What quantity was there of each?

Observe, that after the question is put into equation, the purpose is to make x stand alone in one member of the equation, equal to a known quantity in the other member, then the value of x is found. In the preceding examples in this Art. x has been found only in the first member, but connected with known quantities by the signs + and —. *In the solution of these equations the first thing was to unite all the* x*'s into one term, and all the known quantities into another. Then, if the number which stood on the same side with* x*, had the sign + before it, that number was subtracted from the other member of the equation; but if it had the sign — before it, it was added to the other member. Then the second member was divided by the coefficient of* x*, and the answer was obtained.*

10. A and B began to trade with equal stocks. In the first year A gained a sum equal to twice his stock and £27 over;

B gained a sum equal to his stock and £153 over. Now the amount of both their gains was equal to 5 times the stock of either. What was the stock?

Let x denote the stock. Then A's gain was $2x + 27$, and B's was $x + 153$. These added together must make 5 times the stock, that is, $5x$.

$$5x = 2x + 27 + x + 153$$

Uniting the x's in 2d member, and the numbers,

$$5x = 3x + 180$$

Subtracting $3x$ from both sides,

$$2x = 180$$
$$x = 90$$

11. A young man being asked his age, answered that if the age of his father, which was 44 years, were added to twice his own, the sum would be four times his own age. What was his age?

12. A man meeting some beggars, gave each of them 4 pence, and had 16 pence left; if he had given them 6 pence apiece, he would have wanted 12 pence more for that purpose. How many beggars were there, and how much money had he?

Let x represent the number of beggars.

13. A man has six sons, each of whom is 4 years older than his next younger brother; and the eldest is three times as old as the youngest. Required their ages.

14. Three persons, A, B, and C, make a joint contribution, which in the whole amounts to £76, of which A contributes a certain sum, B contributes as much as A and £10 more, and C as much as A and B both. Required their several contributions.

15. A boy, being sent to market to buy a certain quantity of meat, found that if he bought beef, which was 4 pence per pound, he would lay out all the money he was entrusted with; but if he bought mutton, which was 3½ pence per pound, he would have 2 shillings left. How much meat was he sent for?

16. A man lying at the point of death left all his estate to his three sons, to be divided as follows: to A he gave one half of the whole wanting $500; to B one third; and to C the rest,

which was $100 less than the share of B. What was the whole estate, and what was each son's share?

Let x represent the whole estate.

$$\text{A's share will be } \frac{x}{2}-500$$

$$\text{B's share} \quad . \quad . \quad \frac{x}{3}$$

$$\text{C's share} \quad . \quad . \quad \frac{x}{3}-100$$

These together will be equal to the whole estate, which was represented by x.

$$\frac{x}{2}-500+\frac{x}{3}+\frac{x}{3}-100=x$$

Uniting x's and numbers in the first member,

$$\frac{7x}{6}-600=\frac{6x}{6}$$

$\frac{7x}{6}$ is greater than $\frac{6x}{6}$ by 600, therefore

$$\frac{7x}{6}=\frac{6x}{6}+600$$

$$\frac{x}{6}=600$$

$$x=3600$$

The whole estate is $3600; the shares are $1300, $1200, and $1100, respectively.

17. A father intends by his will, that his three sons shall share his property in the following manner; the eldest is to receive 1000 crowns less than half the whole fortune; the second is to receive 800 crowns less than $\frac{1}{3}$ of the whole; and the third is to receive 600 crowns less than $\frac{1}{4}$ of the whole. Required the amount of the whole fortune, and the share of each.

18. A father leaves four sons, who share his property in the following manner; the first takes 3000 livres less than one half the fortune; the second, 1000 livres less than one third of the whole; the third, exactly one fourth; and the fourth takes 600 livres more than one fifth of the whole. What was the whole fortune, and what did each receive?

19. In a mixture of copper, tin, and lead; 16 lb. less than one half of the whole was copper; 12 lb. less than one third of the whole was tin, and 4 lb. more than one fourth of the whole was lead. What quantity of each was there in the mixture?

20. A general having lost a battle, found that he had only 3600 men more than one half of his army left, fit for action; 600 more than one eighth of them being wounded, and the rest, which amounted to one fifth of the whole army, either slain or taken prisoners. Of how many men did his army consist before the battle?

21. Seven eighths of a certain number exceeds four fifths of it by 6. What is that number?

22. A and B talking of their ages, A says to B, one third of my age exceeds its fourth by 5 years. What was his age?

23. A sum of money is to be divided between two persons, A and B, so that as often as A takes £9, B takes £4. Now it happens that A receives £15 more than B. What is the share of each?

24. In a mixture of wine and cider, 25 gallons more than half the whole was wine, and 5 gallons less than one third of the whole was cider. How many gallons were there of each?

IV. 1. A man having some calves and some sheep, and being asked how many he had of each sort, answered, that he had 20 more sheep than calves, and that three times the number of sheep was equal to seven times the number of calves. How many were there of each?

Let x denote the number of calves.

Then $x + 20$ will denote the number of sheep.

7 times the number of calves is $7x$; 3 times the number of sheep is $3x + 60$; for it is evident that to take 3 times $x + 20$, it is necessary to multiply both terms by 3.

By the conditions these must be equal,

$$7x = 3x + 60.$$

Subtracting $3x$ from both members,

$$4x = 60$$

$$x = 15 = \text{number of calves.}$$

$$x + 20 = 35 = \text{number of sheep.}$$

Ans. 15 calves, and 35 sheep.

2. Two men talking of their ages, the first says, your age is 18 years more than mine, and twice your age is equal to three times mine. Required the age of each.

3. Three men, A, B, and C, make a joint contribution, which in the whole amounts to £276. A contributes a certain sum, B twice as much as A and £12 more, and C three times as much as B and £12 more. Required their several contributions.

4. A man bought 7 oxen and 11 cows for \$591. For the oxen he gave \$15 apiece more than for the cows. How much did he give apiece for each?

Let x denote the price of a cow.

Then the price of an ox will be $x + 15$.

11 cows at x dollars apiece will come to $11\,x$ dollars.

If one ox cost $x + 15$ dollars, 7 oxen will cost 7 times $x + 15$, which is $7\,x + 105$.

The price of the oxen and of the cows added together will make \$591, the whole price.

$$11\,x + 7\,x + 105 = 591$$

Uniting x's, $$18\,x + 105 = 591$$

Subtracting 105 from both members,

$$18\,x = 486$$

Dividing by 18, $$x = 27 = \text{price of cows.}$$

$$x + 15 = 42 = \text{price of oxen.}$$

5. A man bought 20 pears and 7 oranges for 95 cents. For the oranges he gave 2 cents apiece more than for the pears. What did he give apiece for each?

6. A man bought 20 oranges and 25 lemons for \$1.95. For the oranges he gave 3 cents apiece more than for the lemons. What did he give apiece for each?

7. Two persons engage at play, A has 76 guineas, and B 52, before they begin. After a certain number of games lost and won between them, A rises with three times as many guineas as B. How many guineas did A win of B?

Let x denote the number of guineas that A won of B.

Then A, having gained x guineas, will have $76 + x$

B, having lost x guineas, will have only $52 - x$

A has now three times as many as B, that is, 3 times $52 - x$, which is $156 - 3\,x$. It is evident that both 52 and x must be multiplied by 3, because 52 is a number too large by x, therefore 3 times 52 will be too large by $3\,x$.

$$\begin{aligned} 76 + x &= 156 - 3x \\ x &= 156 - 3x - 76 \\ x + 3x &= 156 - 76 \\ 4x &= 156 - 76 \\ 4x &= 80 \\ x &= 20 \end{aligned}$$

Ans. 20 guineas.

Proof. If A won 20 guineas of B, A will have 96 and B 32. 3 times 32 are 96.

This equation is rather more difficult to solve than any of the preceding. In the first place I subtract 76 from both members, so as to remove it from the first member. Then to get $3x$ out of the second member, which is there subtracted, I add $3x$ to both members; then the x's are all in the first member, and the known numbers in the other.

N. B. Any term which has the sign $+$, either expressed or understood, may be removed from one member to the other by giving it the sign $-$; for this is the same as subtracting it from both sides. Thus $x + 3 = 10$; x is not so much as 10 by 3, we therefore say $x = 10 - 3$. Again, $5x = 18 + 3x$. Now $5x$ is more than 18 by $3x$, therefore we may say $5x - 3x = 18$.

Any term which has the sign $-$ before it may be removed from one member to the other by giving it the sign $+$. This is equivalent to adding the number to both sides. Thus $5x - 3 = 17$. In this it appears that $5x$ is more than 17 by 3; therefore we say $5x = 17 + 3$. Again, $5x = 32 - 3x$. Here it appears that $5x$ is not so much as 32 by $3x$; therefore we say $5x + 3x = 32$. This is called *transposition.*

Hence it appears that any term may be transposed from one member to the other, care being taken to change the sign.

In the last example, 76 was *transposed* from the first member to the second, and the sign changed from $+$ to $-$; and $3x$ was *transposed* from the second member to the first, and the sign changed from $-$ to $+$. This has been done in many of the preceding examples.

When a number, consisting of two or more terms, is to be multiplied, all the terms must be multiplied, and their signs preserved. In the last example, $52 - x$, *multiplied by* 3, *gave a product* $156 - 3x$.

8. A person bought two casks of wine, one of which held exactly three times as much as the other. From each he drew

4 gallons, and then there were four times as many gallons remaining in the larger as in the smaller. How many gallons were there in each at first?

Let x denote the number of gallons in the less at first.

Then the number of gallons in the greater will be $3x$.

Taking 4 gallons from each, the less will be $x-4$

And the greater . . . . $3x-4$

The greater is now 4 times as large as the less; 4 times $x-4$ is $4x-16$.

$$4x-16=3x-4$$

By transposing 16, $4x=3x+16-4$

By transposing $3x$, $4x-3x=16-4$

Uniting terms, $x=12=$ less.

$3x=36=$ greater.

Ans. Less 12 gallons, greater 36 gallons.

Proof. 36 is three times 12 according to the conditions. Take 4 from each, then one contains 32 and the other 8. 32 is 4 times 8.

9. A man when he was married was three times as old as his wife; after they had lived together 15 years, he was but twice as old. How old was each when they were married?

10. A farmer has two flocks of sheep, each containing the same number. From one of these he sells 39, and from the other 93; and finds just twice as many remaining in the one as in the other. How many did each flock originally contain?

11. A courier, who travels 60 miles per day, had been despatched 5 days, when a second was sent to overtake him; in order to which, he must go 75 miles per day; in what time will he overtake the former?

12. A and B engaged in trade, A with £240, and B with £96. A lost twice as much as B; and upon settling their accounts it appeared that A had three times as much remaining as B. How much did each lose?

Let x denote B's loss, then $96-x$ will denote what he had remaining. $2x$ will denote A's loss, and $240-2x$ what he had remaining, &c.

13. Two persons began to play with equal sums of money; the first lost 14 shillings, and the other won 14 shillings, and then the second had twice as many shillings as the first. What sum had each at first?

14. Says A to B, I have 5 times as much money as you; yes, says B, but if you will give me $17, I shall have 7 times as much as you. How much had each?

15. Two men, A and B, commenced trade; A had $500 less than 3 times as much money as B; A lost $1500, and B gained $900, then B had twice as much as A. How much had each at first?

16. From each of 15 coins an artist filed the value of 2 shillings, and then offered them in payment for their original value; but being detected, the whole were found to be worth no more than $145. What was their original value?

17. A boy had 41 apples, which he wished to divide between three companions, as follows; to the second he wished to give twice as many as to the first, and three apples more; and to the third he wished to give three times as many as to the second, and 2 apples more. How many must he give to each?

18. A person buys 12 pieces of cloth for 149 crowns: 2 are white, 3 are black, and 7 are blue. A piece of the black costs 2 crowns more than a piece of the white, and a piece of the blue costs 3 crowns more than a piece of the black. Required the price of each kind.

See example 4th of this Art.

19. A man bought 6 barrels of flour and 4 firkins of butter; he gave $2 more for a firkin of butter, than for a barrel of flour; and the butter and flour both cost the same sum. What did he give for each?

20. A grocer sold his brandy for 25 cents a gallon more than his wine, and 37 gallons of his wine came to as much as 32 gallons of his brandy. What was each per gallon?

21. A man bought 7 oxen and 36 cows; he gave $18 apiece more for the oxen than for the cows, and the cows came to three times as much as the oxen wanting $3. What was the price of each?

22. A man sold 20 oranges, some at 4 cents apiece, and some at 5 cents apiece, and the whole amounted to 90 cents. How many were there of each sort?

If he had sold 13 at 5 cents apiece, then the number sold at 4 cents apiece would be 20 — 13, or 7.

In the same manner, if he sold x oranges at 5 cents apiece, then he sold $20 - x$ oranges at 4 cents apiece. x oranges at 5 cents apiece would come to $5x$ cents, and $20 - x$ oranges at 4 cents apiece would come to 4 times $20 - x$ cents, which is $80 - 4x$ cents.

These added together must make 90 cents, therefore

$$5x + 80 - 4x = 90$$

By transposing 80 and uniting terms, $x = 10$ at 5 cents.

Ans. 10 of each sort.

23. A man dying left an estate of $2500 to be divided between his two sons, in such a manner, that twice the elder son's share should be equal to three times the share of the second. Required the share of each.

Let x denote the younger son's share.

Then $2500 - x$ will denote the elder son's share.

Twice the elder son's share is $5000 - 2x$.

By the conditions, $3x = 5000 - 2x$

By transposition, $5x = 5000$

Dividing by 5, $x = 1000$

$$2500 - 1000 = 1500$$

Ans. Elder son $1500, younger son $1000.

24. Two robbers, after plundering a house, found they had 35 guineas between them; and that if one of them had 4 guineas more, he would have twice as many as the other. How many had each?

25. A man sold 45 barrels of flour for $279; some at $5 and some at $8 per barrel. How many barrels were there of each sort?

26. A man sold some oxen and some cows for $330; the whole number was 15. He sold the cows for $17 apiece, and the oxen for $32 apiece. How many were there of each sort?

27. After A had lost 10 guineas to B, he wanted only 8 guineas in order to have as much money as B; and together they had 60 guineas. What money had each at first?

Let x be the number of guineas A had.

Then $60 - x$ will be the number B had.

A lost 10 to B, therefore A's is diminished by 10, and B's increased by 10, which makes A's $x - 10$, and B's $70 - x$.

By the conditions, $x - 10 + 8 = 70 - x$

Transposing and uniting, $2x = 72$

$x = 36 =$ what A had.

$60 - 36 = 24 =$ what B had.

28. Divide the number 197 into two such parts, that four times the greater may exceed five times the less by 50.

29. Two workmen were employed together for 50 days, at 5 shillings per day each. A spent 6 pence a day less than B did, and at the end of the 50 days he found he had saved twice as much as B, and the expense for two days over. What did each spend per day?

Let x denote what A spent per day (in pence).

Then $60 - x$ (5s. being 60d.) will be what he saved per day.

B saved 6d. less than A.

Therefore $54 - x$ will be what B saved per day.

Multiplying both by 50, the number of days,

A saved $3000 - 50x$, and B saved $2700 - 50x$.

By the conditions A saved $2x$ more than twice what B saved.

Therefore $3000 - 50x = 5400 - 100x + 2x$

Transposing and uniting, $48x = 2400$

$x = 50 =$ what A spent.

$50 + 6 = 56 =$ what B spent.

V. 1. Two persons talking of their ages, A said he was 25 years older than B, and that one half of his age was equal to three times that of B wanting 35 years. What was the age of each?

Let x denote the age of B.

Then the age of A will be $x + 25$.

$\frac{1}{2}$ of $x + 25$ is expressed $\dfrac{x+25}{2}$

Hence we have $3x - 35 = \dfrac{x+25}{2}$

Multiplying by 2, $6x - 70 = x + 25$

By transposing x and -70, $6x - x = 25 + 70$

Uniting terms, $5x = 95$

Dividing by 5, $x = 19 =$ B's age.

$x + 25 = 44 =$ A's age.

Note. Since $\frac{1}{2}$ of $x+25$ is $3x-35$, $x+25$ must be twice $3x-35$.

2. Two men talking of their horses, A says to B, my horse is worth \$25 more than yours, and $\frac{3}{5}$ of the value of my horse is equal to $\frac{3}{4}$ of the value of yours. What is the value of each?

Let x denote the value of B's horse.

Then the value of A's will be $x+25$.

$\frac{1}{5}$ of $x+25$ is $\frac{x+25}{5}$, $\frac{3}{5}$ is 3 times as much, that is $\frac{3x+75}{5}$.

By the conditions, $$\frac{3x}{4}=\frac{3x+75}{5}$$

Multiplying by 5, $$\frac{15x}{4}=3x+75$$

Multiplying by 4, $$15x=12x+300$$
$$3x=300$$
$$x=100$$

Ans. A's \$125, B's \$100.

Proof. The first condition is evidently answered. With regard to the second, $\frac{3}{5}$ of 125 is 75, and $\frac{3}{4}$ of 100 is 75.

3. Two men talking of their ages, one says, my age is now $\frac{3}{4}$ of yours, but in twenty years from this time, if we live, it will be $\frac{4}{5}$ of yours. Required the age of each.

Suppose the age of the elder x.

Then the younger will be $\frac{3x}{4}$.

In 20 years the age of the elder will be $x+20$, and of the younger $\frac{3x}{4}+20$.

By the conditions $$\frac{4x+80}{5}=\frac{3x}{4}+20$$

Multiplying by 5, $$4x+80=\frac{15x}{4}+100$$

Multiplying by 4, $$16x+320=15x+400$$

Transposing $15x$, and 320, $$16x-15x=400-320$$
$$x=80=\text{age of elder.}$$
$$\frac{3x}{4}=60=\text{age of younger.}$$

4. A man being asked the value of his horse and chaise, answered, that the chaise was worth $50 more than the horse, and that one half of the value of the horse was equal to one third of the value of the chaise. Required the value of each.

5. Two persons talking of their ages, the first says, $\frac{2}{3}$ of my age is equal to $\frac{3}{7}$ of yours; and the difference of our ages is 10 years. What are their ages?

6. There are two towns situated at unequal distances from Boston, and on the same road. They are 30 miles apart. $\frac{2}{3}$ of the distance of the second from Boston is equal to $\frac{4}{5}$ of the distance of the first. What is the distance of each from Boston?

7. A man being asked the value of his horse and saddle, answered, that his horse was worth $114 more than his saddle, and that $\frac{2}{3}$ of the value of his horse was 7 times the value of his saddle. What was the value of each?

8. A hare is 40 rods before a greyhound, but she can run only $\frac{7}{8}$ as fast as the greyhound. How far will each of them run before the greyhound will overtake the hare?

9. A gentleman paid 4 labourers $136; to the first he paid 3 times as much as to the second wanting $4; to the third one half as much as the first, and $6 more; and to the fourth 4 times as much as to the third, and $5 more. How much did he pay to each?

10. A man bought some cider at $4 per barrel, and some beer at $7. There were 6 barrels more of the cider than of the beer; and $\frac{2}{5}$ of the price of the beer was equal to $\frac{1}{2}$ of the price of the cider. Required the number of barrels of each.

11. Two men commenced trade together; the first put in £40 more than the second, and the stock of the first was to that of the second as 14 to 5. What was the stock of each?

14 to 5 signifies the second is $\frac{5}{14}$ of the first.

12. A man's age when he was married was to that of his wife as 3 to 2; and when they had lived together 4 years, his age was to hers as 7 to 5. What were their ages when they were married?

13. A and B began trade with equal sums of money. In the first year A gained £40, and B lost £40; but in the second, A lost one third of what he then had, and B gained a sum less

by £40 than twice the sum A had lost; when it appeared that B had twice as much money as A. What money did each begin with?

Let x be the number of pounds each had at first. Then $x + 40$ will be the sum A had at the end of the first year; and $x - 40$ the sum B had.

The second year A lost $\frac{1}{3}$ of what he then had, consequently he saved $\frac{2}{3}$; his sum will then be $\frac{2x+80}{3}$.

B gained twice as much as A lost wanting £40; his will be

$$x - 40 + \frac{2x+80}{3} - 40.$$

B had now twice as much as A,

$$\frac{4x+160}{3} = x - 40 + \frac{2x+80}{3} - 40.$$

Multiplying by 3,

$$4x + 160 = 3x - 120 + 2x + 80 - 120.$$

Transposing and uniting,

$$-x = -320.$$

Transposing again, $320 = x$,

Ans. £320.

Note. In this example the result had the sign $-$ in both members, but by transposing it has the sign $+$. It would have been the same thing if the signs had been changed without transposing. The result would have come out right if the first member had been made the second, and the second first, in the first equation.

14. A person playing at cards, cut the pack in such a manner, that $\frac{3}{5}$ of what he cut off were equal to $\frac{3}{8}$ of the remainder. How many did he cut off?

15. Divide $183 between two men, so that $\frac{4}{7}$ of what the first receives, shall be equal to $\frac{3}{10}$ of what the second receives. What will be the share of each?

16. A man sold 20 bushels of grain, rye and wheat; the rye at 5s. and the wheat at 7s. per bushel; $\frac{2}{3}$ of the rye came to as much as $\frac{5}{7}$ of the wheat. How much was there of each?

17. What number is that from which if 5 be subtracted two thirds of the remainder will be 40?

18. A man has a lease for 99 years; and being asked how

much of it was already expired, answered, that two thirds of the time past was equal to four fifths of the time to come. Required the time past, and the time to come.

19. It is required to divide the number 50 into two such parts, that three fourths of one part added to five sixths of the other may make 40.

20. Two workmen received equal sums for their work; but if one of them had received 18 dollars more, and the other 3 dollars less, then $\frac{2}{5}$ of the wages of the latter would have been equal to $\frac{1}{4}$ of the wages of the former. How much did each receive?

21. A certain man, when he married, found that his age was to that of his wife as 7 to 5; if they had been married 8 years sooner, his age would have been to hers as 3 to 2. What were their ages at the time of their marriage?

VI. 1. Divide the number 68 into two such parts, that the difference between the greater and 84, may be equal to three times the excess of 40 above the less.

Let $x =$ the less.

Then $68 - x =$ the greater.

$68 - x$ must be subtracted from 84. Observe that $68 - x$ is not so great as 68 by x. Therefore if I subtract 68 from 84, I shall subtract too much by the quantity x, and I must add x to obtain the true result.

Then we have $84 - 68 + x$ for the difference between 84 and $68 - x$.

The excess of 40 above the less is $40 - x$, and 3 times this is $120 - 3x$.

By the conditions, $84 - 68 + x = 120 - 3x$

Transposing and uniting, $4x = 104$

Dividing by 4, $x = 26 =$ less.

$68 - 26 = 42 =$ greater.

Note. In this question $68 - x$ was subtracted from 84. Instead of x, now put its value, $68 - 26$. Now $68 - 26 = 42$, that is, the number to be subtracted from 84 is 42, and the answer must be 42. When 68 is subtracted from 84, the result is 16, which is too small by 26, the value of x; to this it is necessary to add 26, and it makes 42, the true result, $84 - 68 + 26 = 42$. This shows that we did right in adding x after subtracting 68. This will always be found true. Therefore, *when any of the quantities to be subtracted have the sign — before them, they must be changed to + in subtracting, and those which have + must be changed to —.*

2. A gentleman hired a labourer for 20 days on condition that, for every day he worked, he should receive 7s., but for every day he was idle, he should forfeit 3s. At the end of the time agreed on he received 80 shillings. How many days did he work, and how many days was he idle?

Let $x =$ the number of days he worked.

Then $20 - x =$ the number of days he was idle.

x days, at 7s. a day, would come to $7x$ shillings.

$20 - x$, at 3s. per day, would be $60 - 3x$ shillings. This must be taken out of $7x$.

By the above rule $60 - 3x$, subtracted from $7x$, leaves $7x - 60 + 3x$; for 60 is too much to be subtracted by $3x$.

By the conditions,

$$7x - 60 + 3x = 80.$$

Transposing and uniting,

$$10x = 140.$$

Dividing by 10, $\quad x = 14 =$ days he worked.

$20 - x = 6 =$ days he was idle.

3. Two men, A and B, commenced trade; A had twice as much money as B; A gained \$50, and B lost \$90, then the difference between A's and B's money was equal to three times what B then had. How much did each commence with?

4. Two men, A and B, played together; when they commenced they had \$20 between them, after a certain number of games, A had won \$6, then the excess of A's money above B's was equal to $\frac{2}{3}$ of B's money. How much had each when they commenced?

5. Divide the number 54 into two such parts that the less subtracted from the greater, shall be equal to the greater subtracted from three times the less. What are the parts?

6. It is required to divide the number 204 into two such parts, that $\frac{2}{5}$ of the less being subtracted from the greater, the remainder will be equal to $\frac{3}{7}$ of the greater subtracted from four times the less.

Let $x =$ greater part.

Then $204 - x =$ the less part.

$\frac{2}{5}$ of the less is $\frac{408-2x}{5}$.

By the conditions,

$$x-\frac{408-2x}{5}=816-4x-\frac{3x}{7}.$$

Multiplying by 5,

$$5x-408+2x=4080-20x-\frac{15x}{7}.$$

Multiplying by 7,

$$35x-2856+14x=28560-140x-15x.$$

Transposing and uniting,

$$204x=31416.$$

$$x=154$$

$$204-x=50$$

Let x denote the less number, and solve the question again.

Note. Observe, that after multiplying by 5 in the above example, the signs of both terms of the numerator were changed, that of 408 to —, and that of $2x$ to + ; this was done because it was not required to subtract so much as 408 by $2x$. The change of signs could not be made before multiplying by 5, because the sign — before the fraction showed that the whole fraction was to be subtracted. If the signs of the fraction had been changed at first, it would have been necessary to put the sign + before the fraction. This requires particular attention, because it is of great importance, and there is danger of forgetting it.

7. A man bought a horse and chaise for $341. Now if $\frac{3}{8}$ of the price of the horse be subtracted from twice the price of the chaise, the remainder will be the same as if $\frac{5}{7}$ of the price of the chaise be subtracted from three times the price of the horse. Required the price of each.

8. Two men, A and B, were playing at cards ; when they began, A had only $\frac{3}{5}$ as much money as B. A won of B $23 ; then $\frac{7}{8}$ of B's money, subtracted from A's, would leave one half of what A had at first. How much had each when they began ?

9. A man has a horse and chaise. The horse is worth $44 less than the chaise. If $\frac{4}{7}$ of the value of the horse be subtracted from the value of the chaise, the remainder will be the same as if from the value of the horse you subtract $\frac{2}{3}$ of the ex-

cess of the value of the horse above 84 dollars. What is the value of the horse?

VII. The examples in this article are intended to exercise the learner in putting questions into equation. They require no operations which have not already been explained. It was remarked, that no rule could be given for putting questions into equation, but there is a precept which may be very useful.

Take the unknown quantity, and perform the same operations on it, that it would be necessary to perform on the answer to see if it was right. When this is done the question is in equation.

1. A and B, being at play, severally cut packs of cards so as to take off more than they left. Now it happened that A cut off twice as many as B left, and B cut off seven times as many as A left. How were the cards cut?

Let $x =$ the number B left.
Then $2x =$ the number A cut off.
$52 - x =$ the number B cut off.
$52 - 2x =$ the number A left.

By the conditions, 7 times $52 - 2x$ are equal to $52 - x$.

$$364 - 14x = 52 - x.$$

Take the numbers of the answer and endeavour to prove that they are right, and you will see that you take the same course as above.

2. A man, at a card party, betted 3s. to 2 on every deal. After twenty deals he had won 5 shillings. At how many deals did he win?

Let $x =$ the number of deals he won.
Then $20 - x =$ the number of deals he lost.

Every time he won, he won 2 shillings; that will be $2x$ shillings.

Every loss was 3 shillings; that will be 3 times $20 - x$, or $60 - 3x$.

The loss must be taken from the gain, and he will have 5 shillings left.

$$2x - 60 + 3x = 5.$$

3. What two numbers are to each other as 2 to 3; to each of which, if 4 be added, the sums will be as 5 to 7.

Let $x =$ the first number.

Then $\frac{3x}{2} =$ the second.

Adding 4 to each, they become $x + 4$, and $\frac{3x}{2} + 4$.

The first is now $\frac{5}{7}$ of the second, or the second is $\frac{7}{5}$ of the first.

$$\frac{7x + 28}{5} = \frac{3x}{2} + 4.$$

4. A sum of money was divided between two persons, A and B, so that the share of A was to that of B as 5 to 3. Now A's share exceeded $\frac{5}{9}$ of the whole sum by $50. What was the share of each person?

Let $x =$ A's share.

Then $\frac{3x}{5} =$ B's share.

$$x + \frac{3x}{5} = \text{whole sum.}$$

$$\tfrac{5}{9} \text{ of } x + \frac{3x}{5} \text{ is } \frac{5x}{9} + \frac{15x}{45}, \text{ or } \frac{5x}{9} + \frac{x}{3}$$

By the conditions,

$$x = \frac{5x}{9} + \frac{x}{3} + 50.$$

5. The joint stock of two partners, whose particular shares differed by 48 dollars, was to the lesser as 14 to 5. Required the shares.

6. Four men bought an ox for $43, and agreed that those, who had the hind quarters, should pay $\frac{1}{2}$ cent per pound more than those, who had the fore quarters. A and B had the hind quarters, C and D the fore quarters. A's quarter weighed 158 lb., B's 163 lb., C's 167 lb., and D's 165 lb. What was each per lb., and what did each man pay?

7. A certain person has two silver cups, and only one cover for both. The first cup weighs 12 oz. If the first cup be covered it weighs twice as much as the other cup, but if the second be covered it weighs three times as much as the first. What is the weight of the cover, and of the second cup?

Let $x =$ weight of the cover.
Then $12 + x =$ weight of the first cup covered.
And $6 + \frac{x}{2} =$ weight of the second cup, &c.

8. Some persons agreed to give 6d. each to a waterman for carrying them from London to Gravesend; but with this condition, that for every other person taken in by the way, three pence should be abated in their joint fare. Now the waterman took in three more than a fourth part of the number of the first passengers, in consideration of which he took of them but 5d. each. How many persons were there at first?
Let $x =$ the number of passengers at first.
Then $\frac{x}{4} + 3 =$ the number taken in, &c.

9. Four places are situated in the order of the four letters, A, B, C, D. The distance from A to D is 134 miles, the distance from A to B is to the distance from C to D, as 3 to 2, and one fourth of the distance from A to B, added to half the distance from C to D, is three times the distance from B to C. What are the respective distances?

10. A field of wheat and oats, which contained 20 acres, was put out to a labourer to reap for \$20; the wheat at \$1.20 and the oats \$0.95 per acre. Now the labourer falling ill reaped only the wheat. How much money ought he to receive according to the bargain?

11. Three men, A, B, and C, entered into partnership; A paid in as much as B and one third of C; B paid as much as C and one third of A; and C paid in \$10 and one third of A. What did each pay in?
Let $x =$ the sum A contributed.
Then $\frac{x}{3} + 10 =$ " C "
and $\frac{x}{3} + 10 + \frac{x}{3} =$ " B " &c.

12. A gentleman gave in charity £46; a part of it in equal portions to 5 poor men, and the rest in equal portions to 7 poor women. Now the share of a man and a woman together amounted to £8. What was given to the men, and what to the women?

Let $x =$ the sum a man received.
Then $8 - x =$ the sum a woman received, &c.

13. Suppose that for every 10 sheep a farmer kept, he should plough an acre of land, and should be allowed an acre of pasture for every 4 sheep. How many sheep may that person keep who farms 700 acres?

Let $x =$ the whole number of sheep.

The number of acres ploughed will be $\frac{1}{10}$ of the number of sheep; and the number of acres of the pasture will be $\frac{1}{4}$ of the number of sheep; both these added together must be the whole number of acres, &c.

14. A, B, and C make a joint stock; A puts in $70 more than B, and $90 less than C; and the sum of the shares of A and B is $\frac{4}{5}$ of the sum of the shares of B and C. What did each put in?

Let $x =$ the sum that B put in, &c.

15. Divide the number 85 into two such parts that if the greater be increased by 7 and the less be diminished by 8, they will be to each other in the proportion of 5 to 2.

16. It is required to divide the number 67 into two such parts that the difference between the greater and 75 may be to the excess of the less over 12 in the proportion of 8 to 3.

17. A man bought 12 lemons and a pound of sugar for 56 cents, afterwards he bought 18 lemons and a pound of sugar at the same rate for 74 cents. What was the price of the sugar, and of a lemon?

Let $x =$ the price of the sugar.
Then $56 - x =$ the price of 12 lemons.

And $\frac{56 - x}{12} =$ the price of 1 lemon.

In the same manner,

$$\frac{74 - x}{18} = \text{the price of a lemon.}$$

Hence $\frac{56 - x}{12} = \frac{74 - x}{18}$, &c.

18. A man bought 5 oranges and 7 lemons for 58 cents; afterwards he bought 13 oranges and 6 lemons at the same rate for 102 cents. What was the price of an orange, and of a lemon?

Let $x =$ the price of an orange.

Then $\frac{58 - 5x}{7} =$ the price of a lemon by the first condition, &c.

19. A footman, who contracted for $72 a year and a livery suit, was turned away at the end of 7 months, and received only $32 and the livery. What was the value of the livery?

20. A landlord let his farm for £10 a year in money and a certain number of bushels of corn. When corn sold at 10s. a bushel, he received at the rate of 10s. an acre for his land; but when it sold for 13s. 6d. a bushel, he received 13s. an acre. How many bushels of corn did he receive?

Let $x =$ the number of bushels.

Then $10x + 200 =$ the year's rent in shillings;

$\frac{10x + 200}{10} = x + 20 =$ the number of acres.

$27x + 400 =$ the year's rent at the second rate in sixpences.

$\frac{27x + 400}{26} =$ the number of acres, which must be equal to the other, &c.

21. A man commenced trade with a certain sum of money, which he improved so well, that at the year's end he found he had doubled his first stock wanting $1000; and so he went on every year doubling the last year's stock wanting $1000; at the end of the third year he found that he had just three times as much money as he commenced with. What was his first stock?

22. A man, having a certain sum of money, went to a tavern, where he borrowed as much money as he then had, and then spent a shilling; with the remainder he went to another tavern, where he borrowed as much as he then had, and then spent a shilling, and so he went to a third and a fourth tavern, borrowing and spending as before; after which he had nothing left. How much money had he at first?

23. It is required to divide the number 60 into two such parts, that one seventh of the one may be equal to one eighth of the other.

24. It is required to divide the number 85 into two such parts that $\frac{3}{5}$ of the one added to $\frac{6}{7}$ of the other may make 60.

25. It is required to divide the number 100 into two such parts, that if one third of one part be subtracted from one fourth of the other, the remainder may be 11.

26. It is required to divide the number 48 into two such parts, that one part may be three times as much above 20, as the other wants of 20.

27. A man distributed 20 shillings among 20 people, giving 6 pence apiece to some, and 16 pence apiece to the rest. What number of persons were there of each kind?

28. A man paid £100 with 208 pieces of money, a part guineas at 21s. each, and a part crowns at 5s. each. How many pieces were there of each sort?

29. A countryman had two flocks of sheep, the smaller consisting entirely of ewes, each of which brought him 2 lambs. On counting them he found that the number of lambs was equal to the difference between the two flocks. If all his sheep had been ewes, and brought forth three lambs apiece, his stock would have been 432. Required the number in each flock.

Let $x =$ the number in the less.
Then $2x =$ the number of lambs.
$3x =$ the number in the larger.
$4x =$ the number in both, &c.

30. When the price of a bushel of barley wanted but 3d. to be to the price of a bushel of oats as 8 to 5, four bushels of barley and 7s. 6d. in money were given for nine bushels of oats. What was the price of a bushel of each?

Let $x =$ the price of a bushel of oats in pence.
Then $\frac{8x}{5} - 3 =$ the price of a bushel of barley, &c.

31. A market-woman bought a certain number of eggs at the rate of 2 for a cent, and as many at 3 for a cent, and sold them out at the rate of 5 for two cents; after which she observed, that she had lost four cents by them. How many eggs of each sort had she?

Let x = the number of each sort.

Then $\frac{x}{2}$ = the price of x eggs at 2 for a cent.

And $\frac{x}{3}$ = the price of x eggs at 3 for a cent.

These added together make what the eggs cost.

The whole number is $2x$; these at 5 for two cents come to $\frac{4x}{5}$ cents.

By the conditions, $\frac{x}{2}+\frac{x}{3}=\frac{4x}{5}+4$.

32. A cistern has two fountains to fill it; the first will fill it alone in 7 hours, and the second in 5 hours. In what time will the cistern be filled, if both run together?

Let x = the number of hours required to fill it.

The first would fill $\frac{1}{7}$ of it in an hour, and the second would fill $\frac{1}{5}$ of it in an hour.

Both together then would fill $\frac{1}{7}+\frac{1}{5}$ in an hour; and in x hours both would fill $\frac{x}{7}+\frac{x}{5}$ of it. But by the conditions it was to be filled in x hours.

Therefore, $\frac{x}{7}+\frac{x}{5}=1$ cistern.

33. A gentleman, having a piece of work to do, hired two men and a boy to do it; one man could do it alone in 5 days, the other could do it alone in 8 days, and the boy could do it alone in 10 days. How long would it take the three together to do it?

34. A cistern, into which the water runs by two cocks, A and B, will be filled by them both running together in 12 hours; and by the cock A alone in 20 hours. In what time will it be filled by the cock B alone?

Let x = the time in which B will fill it alone. Both will fill $\frac{1}{12}$ of it in an hour, A alone $\frac{1}{20}$ of it, and B will fill $\frac{1}{12}-\frac{1}{20}$ of it in an hour, &c.

35. A man and his wife usually drank out a vessel of beer in 12 days: but when the man was from home it would usually last the wife alone 30 days. In how many days would the man alone drink it out?

36. The hold of a ship contained 442 gallons of water. This was emptied out by two buckets, the greater of which, holding twice as much as the other, was emptied twice in three minutes, but the less three times in two minutes; and the whole time of emptying was 12 minutes. Required the size of each.

The greater was emptied 8 times in the 12 minutes, &c.

37. Two persons, A and B, have the same income. A saves $\frac{1}{5}$ of his; but B, by spending £80 a year more than A, at the end of 4 years finds himself £220 in debt. What did each receive and expend annually?

38. After paying $\frac{1}{4}$ of my money, and $\frac{1}{5}$ of the remainder, I had 72 guineas left. How much had I at first?

39. A bill of £120 was paid in guineas and moidores, the guineas at 21s., and the moidores at 27s. each; the number of pieces of both sorts was just 100. How many were there of each?

40. It is required to divide the number 26 into three such parts, that if the first be multiplied by 2, the second by 3, and third by 4, the products shall all be equal.

Let $x =$ the first part. The second part must be $\frac{2x}{3}$, and the third part $\frac{2x}{4}$ or $\frac{x}{2}$.

41. It is required to divide the number 54 into three such parts, that $\frac{1}{2}$ of the first, $\frac{1}{3}$ of the second, and $\frac{1}{4}$ of the third, may be all equal to each other.

Let $2x =$ the first part.
Then $3x =$ the second part, &c.

42. A person has two horses and a saddle, which of itself is worth £25. Now if the saddle be put upon the back of the first horse, it will make his value double that of the second; but if it be put upon the back of the second, it will make his value triple that of the first. What is the value of each horse?

43. A man has two horses and a chaise, which is worth $183. Now if the first horse be harnessed to the chaise, the horse and chaise together will be worth once and two sevenths the value of the other; but the other horse being harnessed, the horse and chaise together will be worth once and five

eighths the value of the first. Required the value of each horse.

Equations with two Unknown Quantities.

VIII. Many examples involve two or more unknown quantities. In fact, many of the examples already given involve several unknown quantities, but they were such, that they could all be derived from one. When it is necessary to use two unknown quantities in the solution, the question must always contain two conditions, from which two equations may be derived. When this is not the case the question cannot be solved.

1. A boy bought 2 apples and 3 oranges for 13 cents; he afterwards bought, at the same rate, 3 apples and 5 oranges for 21 cents. How much were the apples and oranges apiece?

Let $x =$ the price of an orange,
and $y =$ the price of an apple.

1. $3x + 2y = 13,$
2. $5x + 3y = 21.$

Multiply the first equation by 3, and the second by 2,

3. $9x + 6y = 39$
4. $10x + 6y = 42.$

Subtract the first from the second, because the y's being alike in each, the difference between the numbers 39 and 42 must depend upon the x's.

5. $x = 3$ cents, the price of an orange.

Putting this value of x into the first equation,

6. $9 + 2y = 13$
7. $y = 2$ cents, the price of an apple.

Proof. 2 apples at 2 cents each come to 4 cents, and 3 oranges at 3 cents come to 9 cents. $9 + 4 = 13$. So 3 apples and 5 oranges come to 21 cents.

Note. In this example I observed, that the coefficient of y in the first equation is 2, and in the second, the coefficient of y is 3. I multiplied the whole of the first equation by 3, and the whole of the second by 2; this formed two new equations in which the coefficients of y are alike. If the first equation had been multiplied by 5 and the second by 3, the coefficients of x would have been alike, and x instead of y would have been made to disappear by subtraction, and the same result would have been finally obtained. It is evident, that the coefficients of either of the unknown quantities may always be rendered alike in the two equations, by multiplying the first equation by the coefficient which the quantity that you wish to make disappear has in the second equation; and the second equation by the coefficient which the same quantity has in the first equation. They may be rendered alike more easily, when they have a common multiple less than their product.

2. A person has two horses, and a saddle which of itself is worth £10; if the first horse be saddled, he will be worth $\frac{6}{7}$ as much as the other, but if the second horse be saddled, he will be worth $\frac{8}{5}$ as much as the first. What is the value of each horse?

A question similar to this has already been solved with one unknown quantity, but it will be more easily solved by using two of them.

Let $x =$ the value of the first horse,
and $y =$ the value of the second horse.

1. By the conditions, $\frac{6y}{7} = x + 10$

2. " $\frac{8x}{5} = y + 10$

3. By transposition, $\frac{6y}{7} - x = 10$

4. " $\frac{8x}{5} - y = 10$

Multiply the 3d by 7, and the 4th by 5, to free them from denominators;

5. $-7x + 6y = 70$
6. $8x - 5y = 50$

Multiply the 5th by 5 and the 6th by 6, in order to make the coefficients of y alike in the two;

7. $-35x + 30y = 350$
8. $48x - 30y = 300$

Add together 7th and 8th,

9. $48x - 35x + 30y - 30y = 350 + 300$
10. Uniting terms, $13x = 650$
11. $x = 50$

Putting 50, the value of x, into the 5th,

12 $6y - 350 = 70$

13. $6y = 420$

14. $y = 70$

Ans. The first is worth £50, and the second £70.

Note. In this example the 30 y in the 7th equation had the sign $+$, and in the 8th the sign $-$ before it, hence it was necessary to add the two equations together in order to make the y disappear, or as it is sometimes called, to *eliminate* y.

3. A market-woman sells to one person, 3 quinces and 4 melons for 25 cents, and to another, 4 quinces and 2 melons, at the same rate, for 20 cents. How much are the quinces and melons apiece?

4. In the market I find I can buy 5 bushels of barley and 6 bushels of oats for 27s., and of the same grain 4 bushels of barley and 3 bushels of oats for 18s. What is the price of each per bushel?

5. My shoemaker sends me a bill of $12 for 1 pair of boots and 3 pair of shoes. Some months afterwards he sends me a bill of $20 for 3 pair of boots and 1 pair of shoes. What are the boots and shoes a pair?

6. Three yards of broadcloth and 4 yards of taffeta cost 57s., and at the same rate 5 yards of broadcloth and 2 yards of taffeta cost 81s. What is the price of a yard of each?

7. A man employs 4 men and 8 boys to labour one day, and pays them 40s.; the next day he hires, at the same wages, 7 men and 6 boys, and pays them 50s. What are the daily wages of each?

8. A vintner sold at one time 20 dozen of port wine and 30 doz. of sherry, and for the whole received £120; and at another time, sold 30 doz. of port and 25 doz. of sherry at the same prices as before, and for the whole received £140. What was the price of a dozen of each sort of wine?

9. A gentleman has two horses and one chaise. The first horse is worth $180. If the first horse be harnessed to the chaise, they will together be worth twice as much as the second horse; but if the second be harnessed, the horse and chaise will be worth twice and one half the value of the first. What is the value of the second horse, and of the chaise?

10. Two men, driving their sheep to market, A says to B, give me one of your sheep and I shall have as many as you; B says to A, give me one of your sheep and I shall have twice as many as you. How many had each?

Let $x =$ the number A had,
And $y =$ the number B had.
If B gives A one, their numbers will be

$$x + 1 \text{ and } y - 1.$$

If A gives B one, their numbers will be

$$x - 1 \text{ and } y + 1, \text{ \&c.}$$

11. If A gives B $5 of his money, B will have twice as much as A has left; but if B gives A $5 of his money, A will have three times as much as B has left. How much has each?

12. A man bought a quantity of rye and wheat for £6, the rye at 4s. and the wheat at 5s. per bushel. He afterwards sold $\frac{1}{5}$ of his rye and $\frac{3}{4}$ of his wheat at the same rate for £2. 17s. How many bushels were there of each?

13. A man bought a cask of wine, and another of gin for $210; the wine at $1.50 a gallon, and the gin at $0.50 a gallon. He afterwards sold $\frac{2}{3}$ of his wine, and $\frac{3}{7}$ of his gin for $150, which was $15 more than it cost him. How many gallons were there in each cask?

14. A countryman, driving a flock of geese and turkeys to market, in order to distinguish his own from any he might meet with on the road, pulled three feathers out of the tail of each turkey, and one out of the tail of each goose, and found that the number of turkeys' feathers exceeded twice those of the geese by 15. Having bought 10 geese and sold 15 turkeys by the way, he was surprised to find that the number of geese exceeded the number of turkeys in the proportion of 7 to 3. Required the number of each at first.

Let $x =$ the number of turkeys,
and $y =$ the number of geese.

1. . . . . $3x = 2y + 15$

2. . . . . $y + 10 = \dfrac{7x - 105}{3}$

3. Freeing the 2d from fractions, $3y + 30 = 7x - 105$

Instead of the method employed above for eliminating one of the unknown quantities, we may find the value of one of them in one equation, as if the other were known; and then this value may be substituted in the other, and an equation will be obtained, containing only one unknown quantity, which may be solved the usual way.

4. Divide the first by 3, $x = \frac{2y + 15}{3}$

5. Multiply the 4th by 7, $7x = \frac{14y + 105}{3}$

Substitute this value of $7x$ in the 3d,

6. $3y + 30 = \frac{14y + 105}{3} - 105$

7. Multiply by 3, $9y + 90 = 14y + 105 - 315$
8. Transposing & uniting, $300 = 5y$
$y = 60.$

The value of x may be found by substituting 60 for y in the 4th,

9. $x = \frac{120 + 15}{3} = 45.$

Ans. 45 turkeys, and 60 geese.

Let the learner go back and solve, in this manner, the preceding examples in this Art. Sometimes one method is preferable and sometimes the other.

15. A person expends $1 in apples and pears, buying his apples at 3 for a cent, and his pears at 2 cents apiece; afterwards he accommodates his neighbour with $\frac{1}{2}$ of his apples and $\frac{1}{4}$ of his pears for 30 cents. How many of each did he buy?

Let $x =$ the number of apples.
And $y =$ the number of pears.
Then $\frac{x}{3} =$ the price of the apples.
And $2y =$ the price of the pears, &c.

16. A market-woman bought eggs, some at the rate of 2 for a cent, and some at the rate of 3 for two cents, to the amount of 65 cents; she afterwards sold them all for 120 cents, and thereby gained one half cent on each egg. How many of each kind did she buy?

17. It is required to find two numbers such, that if $\frac{1}{2}$ of the first be added to the second, the sum will be 30, and if $\frac{1}{3}$ of the second be added to the first, the sum will be 30.

18. It is required to find two numbers such, that $\frac{3}{5}$ of the first and $\frac{2}{7}$ of the second added together will make 12, and if the first be divided by 2 and the second be multiplied by 3, $\frac{2}{3}$ of their sum will be 26.

19. Two persons, A and B, talking of their ages, says A to B, 8 years ago I was three times as old as you were, and 4 years hence I shall be only twice as old as you. Required their present ages.

20. There is a certain fishing rod, consisting of two parts, the upper of which is to the lower as 5 to 7; and 9 times the upper part, together with 13 times the lower part, is equal to 11 times the whole rod and 8 feet over. Required the length of the two parts.

21. A vintner has two kinds of wine, one at 5s. a gallon, and the other at 12s. of which he wishes to make a mixture of 20 gallons, that shall be worth 8s. a gallon. How many gallons of each sort must he use?

22. A vintner has 2 casks of wine, from each of which he draws 8 gallons; and finds that the number of gallons remaining in the less, is to that in the greater as 2 to 5. He then puts 1 gallon of water into the less, and 5 gallons into the greater, and then the quantities are in the proportion of 5 to 13. What quantity did each contain at first?

23. A farmer, after selling 13 sheep and 5 cows, found that the number of sheep he had remaining, was to that of his cows in the proportion of 4 to 3. After three years he found that he had 57 more sheep, and 10 more cows than he had at first; and that the proportions were then as 3 to 1. What number of each had he at first?

24. When wheat was 8 shillings a bushel, and rye 5 shillings, a man wished to fill his sack with a mixture of wheat and rye, for the money he had in his purse. If he bought 15 bushels of wheat, and laid out the rest of his money in rye, he would want 3 bushels to fill his sack; but if he bought 15 bushels of rye, and then filled his sack with wheat, he would have 15 shillings left. How much of each must he purchase in order to lay out his money and fill his sacks?

25. A grocer had 2 casks of wine, the smaller at 7s. per gallon, the larger at 10s. The whole was worth $112. When

he had drawn 10 gals. from each, he mixed the remainder together and added 3¾ gals. of water, and the mixture was worth 8s. per gal. How many gallons of each sort were there at first?

Equations, Generalization.

IX. In the examples hitherto proposed a numerical result has always been obtained. The solution with numbers has been performed at the same time with the reasoning; and when the work was finished, no traces of the operations remained in the result. But algebra has a more important purpose. Pure algebra never gives a numerical result, but is used to trace *general principles* and to form *rules*. In order to preserve the work so that the operations may appear in the result, it will be necessary to introduce a few more signs.

1. It is required to divide $500 between two men, so that one of them may have three times as much as the other.

Let $x =$ the less part.

The equation will be

$$\begin{aligned} x + 3x &= 500 \\ 4x &= 500 \\ x &= 125 \\ 3x &= 375 \end{aligned}$$

Ans. One part is $125, and the other $375.

This question is to divide 500 into two such parts, that one part may be three times as much as the other. It is evident that the process will be the same for any other number, as for 500.

Let the number to be divided be represented by the letter a. This will stand for any number.

Then the question will be, to divide any number, a, into two such parts, that one part may be three times as much as the other.

The equation will be

$$\begin{aligned} x + 3x &= a \\ 4x &= a \\ x &= \frac{a}{4} \\ 3x &= \frac{3a}{4} \end{aligned}$$

The work is now preserved in the result, and it appears that one part will be $\frac{1}{4}$ of the number to be divided; and the other, $\frac{3}{4}$ of it. This is a rule that will apply to any number.

Suppose $a = 500$ as in the example.

Then $\frac{a}{4} = 125$; and $\frac{3a}{4} = 375$.

Ans. One part is $125, and the other $375; the same as above.

Suppose it is required to divide $7532 in the same proportions.

Then $a = 7532$; $\frac{a}{4} = 1883$; and $\frac{3a}{4} = 5649$.

Ans. One part is $1883, and the other is $5649.

2. A man sold some apples, some pears, and some oranges for a number a of cents, the apples at two cents apiece, the pears at three cents apiece, and the oranges at five cents apiece. There were twice as many pears as oranges, and three times as many apples as pears. How many were there of each?

Let $x =$ the number of oranges.
Then $2x =$ the number of pears.
And $6x =$ the number of apples.
By the conditions, $12x + 6x + 5x = a$

$$23x = a$$

$$x = \frac{a}{23} = \text{No. of oranges.}$$

$$2x = \frac{2a}{23} = \text{`` of pears.}$$

$$6x = \frac{6a}{23} = \text{`` of apples.}$$

Suppose $a = 184$ cents, then $\frac{1}{23}$ of $184 = 8 =$ the number of oranges; $2 \times 8 = 16 =$ the number of pears; and $6 \times 8 = 48 =$ the number of apples. This is easily proved. 8 oranges, at 5 cents apiece, come to 40 cents; 16 pears, at 3 cents apiece, come to 48 cents; and 48 apples, at 2 cents apiece, come to 96 cents;

$$40 + 48 + 96 = 184.$$

The learner may be curious to know, how it is possible to make the examples in such a manner, that the answer may always come out a whole number when it is wished; for if the numbers were taken at random, there would frequently be fractions in the result. The method is to solve it first with a letter, as has been done in the two preceding examples. If any number, which is divisible by 4, be put in the place of a, in the first example, the answer will be in whole numbers. And if any number, which is divisible by 23, be put in the place of a, in the second example, the answer will be in whole numbers.

Let the learner now generalize the examples in Art. I., by substituting a letter instead of the number; and after the result is obtained, put in the numbers again, and see if the answers agree. Let him also try other numbers.

The examples in Art. II. may be generalized in the same manner.

3. A man being asked his age, answered, that if its half and its third were added to it, the sum would be 88. Required his age.

Instead of 88 put a, and let $x =$ the number required.

$$x + \frac{x}{2} + \frac{x}{3} = a$$

$$\frac{11\,x}{6} = a$$

$$11\,x = 6\,a$$

$$x = \frac{6\,a}{11}$$

Any number that is divisible by 11, being put in the place of a, will give an answer in whole numbers. Let $a = 88$, then $\frac{6}{11}$ of it is 48, agreeing with the answer in Art. II.

In the course of the solution it appears, that a is equal to $\frac{11}{6}$ of x; and the result shows, that x is equal to $\frac{6}{11}$ of a. That is, the value of x is found by multiplying a by the fraction $\frac{11}{6}$ inverted.

4. In an orchard of fruit-trees, $\frac{1}{3}$ of them bear apples, $\frac{1}{4}$ of them cherries, and the remainder, which is a, bear peaches. How many trees are there in the orchard?

Let $x =$ the whole number of trees.

Then $$x = \frac{x}{3} + \frac{x}{4} + a$$

$$\frac{12x}{12} = \frac{4x}{12} + \frac{3x}{12} + a$$

$$\frac{5x}{12} = a$$

$$5x = 12a$$

$$x = \frac{12a}{5}.$$

Any number that is divisible by 5, may be put in the place of a. If $a = 15$, the answer is 36.

Proof. $$\frac{36}{3} + \frac{36}{4} + 15 = 36.$$

5. The 8th example of Art. II. is solved as follows:

Instead of 100 put a, and let $x =$ the whole number of geese.

Then $$x + x + \frac{x}{2} + 2\tfrac{1}{2} = a$$

Multiplying by 2, $5x + 5 = 2a$

By transposition, $5x = 2a - 5$

$$x = \frac{2a - 5}{5}; \text{ or}$$

$$x = \frac{2a}{5} - \frac{5}{5} = \frac{2a}{5} - 1$$

Let $a = 100$.

Then $$x = \frac{2 \times 100 - 5}{5} = \frac{195}{5} = 39;$$

or $$x = \frac{2 \times 100}{5} - 1 = 40 - 1 = 39.$$

Let $a = 135$, and find the answer in the same way.

The answer will be 53.

Proof. $53 + 53 + 26\tfrac{1}{2} + 2\tfrac{1}{2} = 135$.

The learner may now generalize the examples in Art. II.

The preceding examples admit of being generalized still more, but the process would be too difficult for the learner at present. The following question admits it more easily.

6. (Art. III. Exam. 1.) Two men, A and B, hired a pasture for \$55, and A was to pay \$13 more than B. How much did each pay ?

This question is, to divide the number 55 into two such parts, that one may exceed the other by 13.

*Let us represent 55 by a, and 13 by b. The question now is to divide the number a, into two such parts, that one may exceed the other by the number b : a and b being any two numbers, of which a is the larger.

Let $x =$ the less part.
Then $x + b =$ the greater part.
And $x + x + b = a$
$2x + b = a$
By transposition, $2x = a - b$
Dividing by 2, $x = \frac{a}{2} - \frac{b}{2} = \frac{a - b}{2}$

When a number, consisting of two or more parts, as $a - b$, is to be divided, it is evident that all the terms must be divided, as $\frac{a}{2} - \frac{b}{2}$. But the fractions $\frac{a}{2}$ and $\frac{b}{2}$, having a common denominator, one numerator may be subtracted from the other. Hence $\frac{a}{2} - \frac{b}{2}$ is the same as $\frac{a-b}{2}$. This is easily seen in numbers. See below, where 55 and 13 are substituted for a and b.

Hence it appears, that *the less part is found by subtracting half of the excess of the greater above the less from half the number to be divided ; or by taking half the difference between the number to be divided and the excess.*

The greater part is equal to $x + b$; hence if b be added to $\frac{a}{2} - \frac{b}{2}$ it will give the greater part :

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* Whenever the learner finds any difficulty in comprehending the operations in the general solutions, let him first solve the questions with the numbers.

$$x+b=\frac{a}{2}-\frac{b}{2}+b;$$

or
$$x+b=\frac{a}{2}-\frac{b}{2}+\frac{2b}{2};$$

or
$$x+b=\frac{a}{2}+\frac{b}{2}=\frac{a+b}{2}.$$

The greater is found by adding half the excess to half the number to be divided; or by taking half the sum of the number to be divided and the excess.

In the above example,

$$\text{A's part} = \frac{55}{2}+\frac{13}{2}, \text{ or } \frac{55+13}{2}=34.$$

$$\text{B's part} = \frac{55}{2}-\frac{13}{2}, \text{ or } \frac{55-13}{2}=21.$$

Let the learner generalize this question by making $x =$ the greater part. The same results will be obtained.

This is a general rule, and will apply to all questions like it, and should be remembered, for it is frequently useful.

Let the learner find the answers to the 2d, 3d, and 7th examples of Art. III. by this rule. That is, by putting the numbers of those examples in the place of a and b in the formulas.

It is easy to see the propriety of the rule. For the formula $\frac{a-b}{2}$ or $\frac{55-13}{2}=\frac{42}{2}$, shows, that if the \$13 that A pays more than B, be taken out, the remainder is to be paid in equal parts by them. Also the formula $\frac{a+b}{2}$ or $\frac{55+13}{2}=\frac{68}{2}$, shows, that if B were to pay \$13 more, he would pay as much as A, and the rent would be paid in equal parts by them.

7. A father, who has three sons (Art. III. exam. 4), leaves them 16000 crowns. The will specifies, that the eldest shall have 2000 crowns more than the second, and that the second shall have 1000 crowns more than the third. What is the share of each?

Let a represent the whole number of crowns, b what the eldest son's share exceeds that of the second, and c what the share of the second son exceeds that of the third.

This question may be expressed in general terms, thus: To divide a given number a, into three such parts, that the greatest may exceed the mean by a given number b, and the mean may exceed the least by a given number c

Let $x =$ the greatest.
Then $x - b =$ the mean.
And $x - b - c =$ the least.

By the conditions,

$$x + x - b + x - b - c = a$$
$$3x - 2b - c = a$$

By transposition, $3x = a + 2b + c$

Dividing by 3, $x = \frac{a}{3} + \frac{2b}{3} + \frac{c}{3}$.

Or because the fractions have a common denominator,

$$x = \frac{a + 2b + c}{3}.$$

This is the formula for the greatest part. The mean is $x - b$, or b subtracted from $\frac{a}{3} + \frac{2b}{3} + \frac{c}{3}$, thus;

$$x - b = \frac{a}{3} + \frac{2b}{3} + \frac{c}{3} - b,$$

or $$x - b = \frac{a}{3} + \frac{2b}{3} + \frac{c}{3} - \frac{3b}{3},$$

or $$x - b = \frac{a}{3} - \frac{b}{3} + \frac{c}{3} = \frac{a - b + c}{3}.$$

The least part is $x - b - c$, or c, subtracted from

$$\frac{a}{3} - \frac{b}{3} + \frac{c}{3};$$

$$x - b - c = \frac{a}{3} - \frac{b}{3} + \frac{c}{3} - c,$$

or $$x - b - c = \frac{a}{3} - \frac{b}{3} + \frac{c}{3} - \frac{3c}{3},$$

or $$x - b - c = \frac{a}{3} - \frac{b}{3} - \frac{2c}{3} = \frac{a - b - 2c}{3}.$$

The greatest part is $\frac{a+2b+c}{3}$

The mean do. $\frac{a-b+c}{3}$.

The least do. $\frac{a-b-2c}{3}$.

The eldest son's share, by the first formula, is

$$\frac{16000+2\times 2000+1000}{3}=7000 \text{ crowns.}$$

The other shares may be found by the other two formulas.

Let the learner solve this question by making x equal to the less part, and also by making it equal to the mean.

Exam. 5th, Art. III. may be solved by this formula. Let the learner generalize the questions in Art. III. as far as to Exam. 16th.

The examples in Art. I. may be generalized still farther.

8. A man bought corn at 4s. (a) per bushel, rye at 6s. (b) per bushel, and wheat at 8s. (c) per bushel: there was an equal quantity of each sort. The whole came to 90s. (d). How many bushels were there of each?

It will readily be perceived that it is impossible actually to perform the operations of addition, subtraction, &c. on letters; but it is easy to represent these operations. We however frequently speak of adding, subtracting, multiplying, and dividing algebraic quantities, by which we mean, representing these operations. We have seen that to express 3 times x or 3 times a we write $3x$, $3a$, that is, x or a multiplied by 3. In the same manner, if we wish to express a times x, that is, x multiplied by a, we write ax; and if we wish farther to express that ax (that is, a times x) is to be multiplied by b, we write abx.

*Let $x =$ the number of bushels of each.
Then $ax =$ the price of the corn.
$bx =$ the price of the rye.
And $cx =$ the price of the wheat.

$$ax+bx+cx=d.$$

Here x is taken a times, and b times, and c times, that is, $(a+b+c)$ times. This may be expressed thus, $(a+b+c)\,x$,

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* Let the learner perform this example first by the numbers.

enclosing the three coefficients connected by their signs in a parenthesis.

This will be plain if we put it in numbers.

$4x + 6x + 8x$ is the same as $(4 + 6 + 8)x$, that is, $18x$.

$$(a + b + c)x = d$$

If we had $$18x = d$$

we should divide by 18, $$x = \frac{d}{18}$$

In the same manner divide by $(a + b + c)$,

$$x = \frac{d}{a + b + c}$$

Particular Ans. 5 bushels.

This general formula is expressed in words as follows: Divide the price of the whole by the price of a bushel of each sort added together, and it will give the number of bushels of each sort.

9. A father dying left \$25000 (or a) to be divided between his wife, son, and daughter; his son was to have 3 (or b) times as much as the daughter, and the wife 2 (or c) times as much as the son. What was the share of each?

Let $x =$ the share of the daughter.
Then $3x$ or $bx =$ the share of the son.
And $6x$ or $bcx =$ the share of the wife.

$$x + 3x + 6x = 25000$$
$$x + bx + bcx = a$$
$$(1 + 3 + 6)x = 10x = 25000$$
$$(1 + b + bc)x = a$$
$$x = \frac{25000}{10} = 2500$$
$$x = \frac{a}{1 + b + bc}.$$

In this example observe that x is taken 1 time, and b times, and bc times. When a letter is written without a coefficient, it is always understood to have 1 for its coefficient; thus x is the same as $1x$.

Having found the share of the daughter, it is easy to find the shares of the other two.

The son's share is $3x = 7500$, or $bx = \dfrac{ab}{1+b+bc}$.

The wife's do. is $6x = 15000$, or $bcx = \dfrac{abc}{1+b+bc}$.

The learner may now generalize some of the examples in Art. I. in this manner.

10. A gentleman, distributing some money among some beggars, found, that in order to give them 8 (or a) cents apiece, he should want 5 (or b) cents; he therefore gave them 7 (or c) cents, and he had 4 (or d) cents left. How many beggars were there?

Let $x =$ the number of beggars.

Then

$$8x - 5 = 7x + 4$$

or

$$ax - b = cx + d$$

$$8x - 7x = 5 + 4 = 9$$

$$ax - cx = b + d$$

$$(8-7)x = 9$$

$$(a-c)x = b + d$$

$$x = 9$$

$$x = \frac{b+d}{a-c}.$$

Particular Ans. 9 beggars.

General Ans. $\dfrac{b+d}{a-c}$.

11. There is a cistern which is supplied by two pipes; the first will fill it alone in 7 (or a) hours, the second will fill it alone in 5 (or b) hours. In what time will it be filled if both run together?

Let $x =$ the number of hours in which both together will fill it.

The first will fill $\frac{1}{7}$ or $\frac{1}{a}$ of it in one hour, and the second will fill $\frac{1}{5}$ or $\frac{1}{b}$ of it in one hour; both together will fill $\frac{1}{7} + \frac{1}{5}$ or $\frac{1}{a} + \frac{1}{b}$ of it in one hour. In x hours they will fill x times as much, that is,

$$\frac{x}{7} + \frac{x}{5}, \text{ or } \frac{x}{a} + \frac{x}{b}.$$

But x hours is the whole time, therefore, the cistern being 1,

$$\frac{x}{7}+\frac{x}{5}=1, \text{ or } \frac{x}{a}+\frac{x}{b}=1.$$

Clearing of fractions,

$$5\,x+7\,x=35 \qquad b\,x+a\,x=a\,b$$

Uniting coefficients, $12\,x=35 \qquad (b+a)\,x=a\,b$

$$x=2\tfrac{11}{12} \qquad x=\frac{a\,b}{a+b}.$$

Particular Ans. $2\frac{11}{12}$ hours.

General Ans. $\frac{a\,b}{a+b}$.

Suppose one pipe would fill the cistern in $8\frac{1}{2}$ hours, and the other in $4\frac{3}{4}$ hours, and find the answer by the general formula.

Ans. $3\frac{5}{106}$ hours.

12. Suppose it were required to make a rule for Fellowship. First take a particular case.

Three men, commencing trade together, furnished money in the following proportions; A \$8 as often as B \$5, and as often as C \$3. They gained \$800. What is each man's share of the gain?

It is evident that they must receive in the proportion of the capital that they respectively furnished.

Let $x=$ A's share of the gain.

Then $\frac{5x}{8}=$ B's share.

And $\frac{3x}{8}=$ C's share.*

$$x+\frac{5\,x}{8}+\frac{3\,x}{8}=800$$

$$8\,x+5\,x+3\,x=6400$$

$$16\,x=6400$$

$$x=400=\text{A's share.}$$

$$\frac{5\,x}{8}=250=\text{B's share.}$$

$$\frac{3\,x}{8}=150=\text{C's share.}$$

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* See Art. II. Examp. 24 and 25.

Now, instead of 8, 5, and 3, suppose they furnished in the proportion of m, n, and p; and let the whole gain be a.

Let $x =$ A's share of the gain.

Then $\frac{n\,x}{m} =$ B's share.

And $\frac{p\,x}{m} =$ C's share.

Then we have

$$x + \frac{n\,x}{m} + \frac{p\,x}{m} = a$$

$$m\,x + n\,x + p\,x = m\,a$$

$$(m + n + p)\,x = m\,a$$

$$x = \frac{m\,a}{m+n+p} = \text{A's share.}$$

B's share is $\frac{n\,x}{m}$, or the $\frac{n}{m}$ part of $\frac{m\,a}{m+n+p} =$ A's share.

Since a fraction is divided by dividing its numerator, the $\frac{1}{m}$ part of $\frac{m\,a}{m+n+p}$, will be found by dividing the numerator $m\,a$ by m. a multiplied by m is $m\,a$, therefore, $m\,a$ divided by m is a. Hence the $\frac{1}{m}$ part of $\frac{m\,a}{m+n+p}$ is $\frac{a}{m+n+p}$, and the $\frac{n}{m}$ part is n times as much, that is $\frac{n\,a}{m+n+p}$, which is B's share.

C's share is $\frac{p\,x}{m}$, or the $\frac{p}{m}$ part of $\frac{m\,a}{m+n+p}$, which is $\frac{p\,a}{m+n+p}$.

A's share is $\frac{m\,a}{m+n+p}$; B's do. $\frac{n\,a}{m+n+p}$; and C's do. $\frac{p\,a}{m+n+p}$.

Hence to find the share of either, *multiply the whole sum to be divided, by the proportion of the stock which he furnished, and divide the product by the sum of their proportions.*

The propriety of this rule is easily seen. For, putting in the numbers instead of the letters, A's share is $\frac{8}{8+5+3}$ or $\frac{8}{16}$ of $800, B's share is $\frac{5}{8+5+3}$ or $\frac{5}{16}$ of it, and C's share is $\frac{3}{8+5+3}$ or $\frac{3}{16}$ of it. That is, the sum of all their proportions is 16, and of these A furnished 8; B, 5; and C, 3.

13. Let it be required to find what sum, put at interest at a given rate, will amount to a given sum in a given time; that is, to find a rule, by which the principal may be found, when the rate, time, and amount are given.

First take a particular case.

A man lent some money for 3 years, interest at 6 per cent, and received for interest and principal $472. What was the sum lent?

Let $x =$ the sum lent.

Then $\frac{6x}{100} =$ the interest for 1 year.

And $\frac{18x}{100} =$ do. for 3 years.

And $x + \frac{18x}{100} =$ the amount for 3 years.

Hence we have
$$x + \frac{18x}{100} = 472$$
$$100x + 18x = 47200$$
$$118x = 47200$$
$$x = \$400 = \text{The sum lent.}$$

It is a custom established among mathematicians to use the first letters of the alphabet for known quantities, and some of the last letters for unknown quantities. It is, however, frequently convenient to choose letters, that are the initials of the words for which they stand, whether the quantities be known or unknown.

To generalise the above example,

Let $p =$ the principal, or sum lent.
$r =$ the rate per annum, which in the above case is $\frac{6}{100}$ or .06.
and $t =$ the time for which it was lent,
and $a =$ the amount.
Then $rp =$ the interest for one year,
and $trp =$ do. for t years,
and $p + trp =$ the amount.

Hence we have
$$p + trp = a$$
$$(1 + tr)p = a$$
$$p = \frac{a}{1 + tr}.$$

That is, *multiply the rate by the time, add 1 to the product, and divide the amount by this, and it will give the principal.*

In the above example the rate is .06, which, multiplied by 3 (the time), gives .18, and one added to this makes 1.18; 472 divided by 1.18 gives 400, as before.

Apply this rule to the following example.

A man owes \$275, due two years and three months hence, without interest. What ought he to pay now, supposing money to be worth $4\frac{1}{2}$ per cent. per annum?

N. B. 2 years and 3 months is $2\frac{1}{4}$ years.

Ans. $\$249\frac{78875}{110125}$.

See Arithmetic, page 84.

The learner may now make rules for the following purposes:

14. The interest, time, and rate being given, to find the principal.

15. The amount, time, and principal being given, to find the rate.

16. The amount, principal, and rate given, to find the time.

17. A man agreed to carry 20 (or a) earthen vessels to a certain place, on this condition; that for every one delivered safe he should receive 8 (or b) cents, and for every one he broke, he should forfeit 12 (or c) cents; he received 100 (or d) cents. How many did he break?

Let $x =$ the number unbroken.

Then $20 - x$ or $a - x =$ the number broken.

For every one unbroken he was to receive 8 or b cents, these will amount to $8x$ or bx; and for every one broken he was to pay back 12 or c cents, these will amount to $240 - 12x$ cents, or $ac - cx$; this must be subtracted from the former.

$240 - 12x$, subtracted from $8x$, is

$$8x - 240 + 12x, \text{ or } 20x - 240.$$

Also $ca - cx$ subtracted from bx, is $bx - ca + cx$; for the quantity $ca - cx$ is not so large as ca, by the quantity cx, therefore when we subtract ca from bx, we subtract too much by cx, and in order to obtain a correct result, it is necessary to add cx.

The equation is

$$\begin{array}{rcr} 20x - 240 = 100 & \text{or} & bx + cx - ac = d \\ 20x = 340 & \text{``} & bx + cx = d + ac \\ & & (b+c)x = d + ac \\ x = 17 & \text{``} & x = \dfrac{d + ac}{b + c}. \end{array}$$

Particular Ans. 17 unbroken, and 3 broken.

General Ans. Unbroken $\dfrac{d + ac}{b + c}$.

Putting numbers into the general answer,

$$\frac{100 + 12 \times 20}{8 + 12} = 17.$$

The propriety of this answer may be shown as follows: If he had broken the whole 20 (or a) he must have paid $12 \times 20 = 240$ (or ac) cents; but instead of paying this, he received 100 (or d) cents. Now the difference to him between paying 240 and receiving 100 is evidently 340, (or $d + ac$) cents. The difference for each vessel between *paying* 12 and *receiving* 8 is 20 (or $b + c$) cents; 340 divided by 20 gives 17, the answer.

The above is a good illustration of *positive* and *negative* quantities, or quantities affected with the signs + and —. The sign + is placed before the quantities, which he is to receive, and the sign — before his losses. We observed that the difference between receiving 100 and losing 240 is 340, that is, the difference between + 100 and — 240 is 340, or their sum. Also the difference between $+d$ and $-ac$ is $d + ac$. So the difference between $+ 8$ and $- 12$ is 20, or between $+ b$ and $- c$ is $b + c$.

Hence it follows, *that to subtract a quantity which has the sign —, we must give it the opposite sign, that is, it must be added.*

X. The learner, by this time, must have some idea of the use of letters, or general symbols, in algebraic reasoning. It has been already observed that, strictly speaking, we cannot actually perform the four fundamental operations on these quantities, as we do in arithmetic; yet in expressing these operations, it is frequently necessary to perform operations so analogous to them, that they may with propriety be called by the same names. Most of these have already been explained; but in order to impress them more firmly on the mind of the learner, they will be briefly recapitulated, and some others explained which could not be introduced before.

Note. Algebraic quantities, which consist of only one term, are called *simple quantities*, as $+ 2\, a$, $- 3\, a\, b$, &c.; quantities which consist of two terms are called *binomials*, as $a + b$, $a - b$, $3\, b + 2\, c$, &c.; those which consist of three terms are called *trinomials;* and in general those which consist of many terms are called *polynomials.*

Simple Quantities.

The addition of simple quantities is performed by writing them after each other with the sign + between them. To express that a is added to b, we write $a + b$. To express that a, b, c, d, and e are added together, we write $a + b + c + d + e$. It is evidently unimportant which term is written first, for $3 + 5 + 8$ is the same as $5 + 3 + 8$, or as $8 + 5 + 3$. So $a + b + c$ has the same value as $b + a + c$.

It has been remarked (Art. I.) that $x + x + x$ may be written $3\, x$. This is *multiplication;* and it arises, as was observed in Arithmetic, Art. III., from the successive addition of the same quantity. $3\, x$, it appears, signifies 3 times the quantity x, that is, x multiplied by 3. So $b + b + b + b + b$ may be written $5\, b$. In the same manner, if x is to be repeated, any number of times, for instance as many times as there are units in a, we write $a\, x$, which signifies a times x, or x multiplied by a.

N. B. The learner should constantly bear in mind that the letters, a, b, c, &c. may be used to represent any known number; or they may be used indefinitely, and any number may afterwards be substituted in their place.

Again, $ab + ab + ab$ may be written $3ab$, that is, 3 times the product ab; also c times the product ab may be written cab.

It may be remarked that a times b is the same as b times a; for a times 1 is a, and a times b must be b times as much, that is, b times a. Hence the product of a and b may be written either ab or ba. In the same manner it may be shown that the product cab is the same as abc. Suppose $a = 3$, $b = 5$, and $c = 2$, then $abc = 3 \times 5 \times 2$, and $cab = 2 \times 3 \times 5$. In fact it has been shown, in Arith. Art. IV., that when a product is to consist of several factors, it is not important in what order those factors are multiplied together. The product of a, b, c, d, e, and f, is written $abcdef$. They may be written in any other order, as $acdbef$, or $fbedca$, but it is generally more convenient to write them in the order they stand in the alphabet.

Let it be required to multiply $3ab$ by $2cd$. The product is $6abcd$; for d times $3ab$ is $3abd$, but cd times $3ab$ is c times as much, or $3abcd$, and $2cd$ times $3ab$ must be twice as much as the latter, that is, $6abcd$.

Hence, *the product of any two or more simple quantities must consist of all the letters of each quantity, and the product of the coefficients of the quantities.*

N. B. Though the product of literal quantities is expressed by writing them together without the sign of multiplication, the same cannot be done with figures, because their value depends upon the place in which they stand. $3ab$ multiplied by $2cd$, for instance, cannot be written $32abcd$. If it is required to express the multiplication of the figures as well as of the letters, they must be written $3ab2dc$, or $3 \times 2abcd$, or $3.2abcd$. That is, the figures must either be separated by the letters or by the sign of multiplication.

Examples in Multiplication.

1. Multiply $3ab$ by $4cdf$. *Ans.* $12abcdf$.
2. $5bcd$ by abc. *Ans.* $5abbccd$.
3. $9egh$ by 8.
4. $13ac$ by $7aacd$.
5. $35abc$ by $13abbd$.

6. Multiply 138 by $5acd$.
7. $25x$ by $11abx$.
8. $42ayy$ by $12xxy$.

It frequently happens, as in some of the above examples, that a quantity is multiplied several times by itself, or enters several times as a factor into a product; as $3aaabb$, into which a enters three times and b twice as a factor. In cases like this the expression may be very much abridged by writing it thus, $3a^3b^2$. That is, by placing a figure a little above the letter, and a little to the right of it, to show how many times that letter is a factor in the product. The figure 3 over the a shows, that a enters three times as a factor; and the 2 over the b, that b enters twice as a factor, and the expression is to be understood the same as $3aaabb$. The figure written over the letter in this manner is called the *index* or *exponent* of that letter. The exponent affects no letter except the one over which it is written.

Care must be taken not to confound exponents with coefficients. The quantities $3a$ and a^3 have very different values. Suppose $a = 4$, then $3a = 12$; whereas $a^3 = 4 \times 4 \times 4 = 64$. In the product $3a^3b^2$ suppose $a = 4$ and $b = 5$, then

$$3a^3b^2 = 3 \times 4 \times 4 \times 4 \times 5 \times 5 = 4800.$$

The expression a^2 is called the *second power* of a, a^3 is called the *third power*, a^4 the *fourth power*, &c. To preserve a uniformity, a, without an exponent, is considered the same as a^1, which is called the first power of a.*

Figures as well as letters may have exponents.

The first power of 3 is written $3^1 = 3$
the second power $3^2 = 3 \times 3 = 9$
the third power $3^3 = 3 \times 3 \times 3 = 27$
the fourth power $3^4 = 3 \times 3 \times 3 \times 3 = 81$
the fifth power $3^5 = 3 \times 3 \times 3 \times 3 \times 3 = 243$.

The multiplication of quantities in which some of the factors are above the first power, is performed in the same manner as in other cases, by writing the letters of both quantities together, taking care to give them their proper exponents. $2\,a\,m^2 \times 3\,c^2\,d^2$ is the same as $2\,a\,m\,m \times 3\,c\,c\,d\,d$, which gives

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* In most treatises on algebra a^2 is called the *square* of a, and a^3 the *cube* of a. The terms *square* and *cube* were borrowed from geometry, but as they are not only inappropriate, but convey ideas very foreign to the present subject, it has been thought best to discard them entirely.

$$6\,a\,m\,m\,c\,c\,d\,d = 6\,a\,m^2\,c^2\,d^2.$$

a^3 multiplied by a^2 gives $a^3\,a^2$; but $a^3 = a\,a\,a$ and $a^2 = a\,a$; hence $a^3\,a^2 = a\,a\,a\,a\,a = a^5$. In all cases the product consists of all the factors of the multiplicand and multiplier. In the last example a is three times a factor in the one quantity, and twice in the other; hence it will be five times a factor in the product. The exponents show how many times a letter is a factor in any quantity; *hence if any letter is contained as a factor one or more times in both multiplier and multiplicand, the exponents being added together will give the exponent of that letter in the product.*

$$a \times a = a^1 \times a^1 = a^{1+1} = a^2. \quad a^2 \times a^1 = a^{2+1} = a^3.$$
$$a^3 \times a^2 = a^{3+2} = a^5, \text{ \&c.}$$

9. Multiply $a^2\,b^2$ by $a\,b^2$. *Ans.* $a^3\,b^4$.
10. $a\,b^5\,c$ by $a^3\,b\,c^2$.
11. $6\,a^5\,c\,d^5$ by $a\,b^5\,c^3$.
12. $a^3\,c^2$ by $a^7\,b^2\,c$.
13. $7\,a^4\,x^3\,y$ by $5\,a^2\,b\,c\,x^2\,y^2$.
14. $17\,b^5\,d^3\,e$ by $4\,b\,b\,c\,d\,e\,e$.
15. $23\,a^2\,x^6$ by $2\,a\,a\,b\,x\,x$.
16. $18\,a\,a\,y\,y$ by $6\,a^3\,y\,y\,x$.

It has already been remarked that the addition of two or more quantities is performed by writing the quantities after each other with the sign + between them. The sum of $3\,a\,b$, $2\,a\,c\,d$, $5\,a^2\,b$, $4\,a\,b$, and $3\,a^2\,b$, is $3\,a\,b + 2\,a\,c\,d + 5\,a^2\,b + 4\,a\,b + 3\,a^2\,b$. But a reduction may be made in this expression, for $3\,a\,b + 4\,a\,b$ is the same as $7\,a\,b$; and $5\,a^2\,b + 3\,a^2\,b$ is the same as $8\,a^2\,b$; hence the expression becomes

$$7\,a\,b + 2\,a\,c\,d + 8\,a^2\,b.$$

Reductions of this kind may always be made when two or more of the terms are *similar*. When two or more terms are composed of the same letters, the letters being severally of the same powers, they are said to be *similar*. The numerical coefficients are not regarded. The quantities $4\,a\,b$ and $3\,a\,b$ are similar, and so are $5\,a^2\,b$ and $3\,a^2\,b$; but $4\,a\,b$ and $5\,a^2\,b$ are not similar quantities, and cannot be united.

The subtraction of algebraic quantities is performed by writing those, which are to be subtracted, after those from which they are to be taken, with the sign — between them.

If b is to be subtracted from a it is written $a - b$. $5\,a\,b^2$ to be subtracted from $8\,a\,b^2$, is written $8\,a\,b^2 - 5\,a\,b^2$. This last expression may be reduced to $3\,a\,b^2$. In all cases when the quantities are *similar*, the subtraction may be performed immediately upon the coefficients.

Compound Quantities.

XI. The addition and subtraction of simple quantities, produce quantities consisting of two or more terms which are called *compound quantities.* $2\,a + c\,d - 3\,b$ is a compound quantity.

Addition of Compound Quantities.

The addition of two or more compound quantities, when all the terms are affected with the sign $+$ will evidently be the same, as if it were required to add together all the simple quantities of which they are composed; that is, they must be written one after the other with the sign $+$ before all the terms except the first. The sum of the quantities $3\,a + 2\,c$ and $b + 2\,d$ is $3\,a + 2\,c + b + 2\,d$.

If the quantities $3\,a\,b + 5\,d$ and $b - c$ be added, in which some of the terms have the sign $-$, the sum will be $3\,a\,b + 5\,d + b - c$; for $b - c$ is less than b, therefore, if b be added the sum will be too large by the quantity c. Hence c must be subtracted from the result.

This may be illustrated by figures. Add together $17 + 10$ and $20 - 6$. Now $20 - 6$ is 14
and $17 + 10 + 20 - 6$ is equal to $17 + 10 + 14$.

From the above observations we derive the following rule for the addition of compound quantities.

Write the quantities after each other without changing their signs, observing that terms which have no sign before them are understood to have the sign $+$.

A sign affects no term except the one immediately before which it is placed; hence it is unimportant in what order the terms are written, for $14 - 5 + 2$ has the same value as $14 + 2 - 5$ or as $-5 + 2 + 14$. Those which have the sign $+$ are to be added together, and those which have the sign $-$ are to be subtracted from their sum. If the first term has the sign

$+$, the sign may be omitted before this term, but the sign $-$ must always be expressed. Great care is requisite in the use of the signs, for an error in the sign makes an error in the result of twice the quantity before which it is written.

Add together $3a + 2bc^2 - 3c^4$
and $5a - 3bc^2 + 2c^4$
and $7ab + 4bc^2 - 8c^4$
and $-a + 3c^4 - 2bc^2$.

The sum is

$$3a + 2bc^2 - 3c^4 + 5a - 3bc^2 + 2c^4 + 7ab + 4bc^2 - 8c^4 - a + 3c^4 - 2bc^2.$$

But this expression may be reduced.

$$3a + 5a - a = 8a - a = 7a,$$

and

$$2bc^2 - 3bc^2 + 4bc^2 - 2bc^2 = 6bc^2 - 5bc^2 = bc^2,$$

and

$$-3c^4 + 2c^4 - 8c^4 + 3c^4 = -11c^4 + 5c^4 = -6c^4;$$

hence the above quantity becomes

$$7a + bc^2 + 7ab - 6c^4.$$

To reduce an algebraic expression to the least number of terms, *collect together all the similar terms affected with the sign* $+$ *and also those affected with the sign* $-$, *and add the coefficients of each separately; take the difference of the two sums and put it into the general result, giving it the sign of the larger quantity.*

Examples in Addition.

1. Add together the following quantities.

$5ab - 2a^2m$
and $3ab - 5am + 2am$.

2. Add together the following quantities.

$13an^2 - 6m + x^2$,
and $7bm - 3x^2 - 8y$,
and $4an^2 + 5ax^2 - 4y$.

3. Add together the following quantities.

$7mab - 16 - 43my$,
and $19acb - 13amb + 37may + 48$,
and $14my - 19may + nb - nx$,
and $4nx - 3bn + 23amy - nb$.

4. Add together the following quantities.

$xy - ax - ay + axy$,
and $-2xy - 2ay + 3ax + 15$,
and $18arx - 73 + 13axy - am$,
and $-15axy - 13am + 43 + 18arx$,
and $arx - 18 + ay - 2axy + 3am$.

5. Add together the following quantities.

$13ax - 2bx - 7$,
and $15bx - 17bxy + 16$,
and $47acd - x$,
and $37 - bx - 2a + 43byx$,
and $acd + byx - 13a$.

Subtraction of Compound Quantities.

XII. The subtraction of simple quantities, as has already been observed, is performed by giving the sign — to the quantity to be subtracted, and writing it before or after the quantity, from which it is to be taken. If it is required to subtract $c + d$ from $a + b$ it is plain that the result will be $a + b - c - d$, for the compound quantity $c + d$ is made up of the simple quantities c and d, which being subtracted separately would give the above result.

From 22 subtract $13 - 7$.

$13 - 7 = 6$.
and $22 - 6 = 16$.

The result then must be 16. But to perform the operation on the numbers as they stand, first subtract 13, which gives $22 - 13 = 9$. This is too small by 7, because the number 13 is larger by 7 than the number to be subtracted, therefore in order to obtain a correct result the 7 must be added; thus $22 - 13 + 7 = 16$, as required.

From a subtract $b - c$.

First subtract b, which gives $a - b$.

This quantity is too small by c because b is larger than $b - c$ by the quantity c. Hence to obtain a correct result c must be added, thus $a - b + c$.

This reasoning will apply to all cases, for the terms affected with the sign $-$ in the quantity to be subtracted diminish that quantity; hence if all the terms affected with $+$ be subtracted, the result will be too small by the quantities affected with $-$, these quantities must therefore be added. The reductions may be made in the result, in the same manner as in addition. Hence the general

RULE. *Change all the signs in the number to be subtracted, the signs $+$ to $-$, and the signs $-$ to $+$, and then proceed as in addition.*

Examples in Subtraction.

1. From $a^2 x + 3 b y - 5 a c^3 - 16$
   Subtract $3 a^2 x + b y - 2 a c^3 - 22$

Operation.

$$\begin{array}{r} a^2 x + 3 b y - 5 a c^3 - 16 \\ - 3 a^2 x - b y + 2 a c^3 + 22 \\ \hline - 2 a^2 x + 2 b y - 3 a c^3 + 6 \end{array}$$

2. From $3 b x^2 - 7 a x^3 + 13$
   Subtract $13 b c - 3 a x^3 - 8$.
   Ans. $3 b x^2 - 13 b c - 4 a x^3 + 21$.

3. From $17 a^2 y + 13 a y^2 - a - 3$
   Subtract $2 a^2 y - b - 11 a + 5$.

4. From $42 a x y - 4 a x$
   Subtract $17 a x - 2 a x y - 5$.

5. From $143 - 17 y$
   Subtract $33 + 4 y - 16 a b$.

6. From $a + 3abc - 1$
   Subtract $1 + 3abc - a$.
7. From $3abz + 2ab - 7z$
   Subtract $2ab - 7z - 2abz$.

Multiplication of Compound Quantities.

XIII. Multiplication of compound quantities is sometimes expressed without being performed. To express that $a + b$ is to be multiplied by $c - d$, it may be written $\overline{a+b} \times \overline{c-d}$ with a *vinculum* over each quantity, and the sign of multiplication between them; or they may be each enclosed in a parenthesis and written together, with or without the sign of multiplication; thus $(a + b) \times (c - d)$ or $(a + b)\,(c - d)$. In the expression $a + b\,(c - d)$, b only is to be multiplied by $c - d$.

Multiply $a + b$ by c.

It is evident that the whole product must consist of the product of each of the parts by c.

$$\begin{array}{lcl} a+b & \qquad 20+4 & = 24 \\ c & \qquad 3 & \quad\;\; 3 \\ \hline ac+bc & \qquad 60+12 & = 72 \end{array}$$

Examples.

1. Multiply $3ab + 2cd$ by ef.
   Ans. $3abef + 2cdef$.
2. Multiply $5ac + bc + 3cd$ by $2e$.
   Ans. $10ace + 2bce + 6cde$.
3. Multiply $6a^2b + b^2c^3$ by $3ab^2$.
4. Multiply $bc^2d^5 + 52a^2b^2 + 13b^3c^2d$
   by $7a^2b^3c$.
5. Multiply $2abd + 3abx^2 + abx^2$.
   by $3abx^2$.
6. Multiply $ax^3 + 3abx^2$ by $13ab^2x^3$

When some of the terms of the multiplicand have the sign — they must retain the same sign in the product.

7. 8. Multiply $a - b$ by c, also $23 - 5$ by 4.

$$\begin{array}{l}
a - b \\
c \\
\hline
a c - b c.
\end{array}
\qquad\qquad
\begin{array}{ccc}
23 - 5 & = & 18 \\
4 & & 4 \\
\hline
92 - 20 & = & 72.
\end{array}$$

Since the quantity $a - b$ is smaller than a by the quantity b, the product $a c$ will be too large by the quantity $b c$. This quantity must therefore be subtracted from $a c$.

9. Multiply $3 a b^2 - c$ by $2 d$.
10. " $2 a d + b d - 3 c$ by $5 a b$.
11. " $3 b c d - e f - 2 a c$ by $5 a c$.
12. " $2 a^3 b e - 5 a^2 + b^2$ by $4 a^3 b^6$.
13. " $17 a c d - 1 + 5 a^2 x - a b^3 x$ by $a^3 c d$.

When both multiplicand and multiplier consist of several terms, each term of the multiplicand must be multiplied by each term of the multiplier.

14. Multiply $12 + 5$ by $7 + 4$.

$$\begin{array}{r}
12 + 5 = 17 \\
7 + 4 = 11 \\
\hline
84 + 35 \qquad \\
+ 48 + 20 \qquad \\
\hline
84 + 35 + 48 + 20 = 187
\end{array}$$

15. Multiply $a + b$ by $c + d$.

$$\begin{array}{l}
a + b \\
c + d \\
\hline
a c + b c + a d + b d.
\end{array}$$

It is evident that if $a + b$ be taken c times and then d times, and the products added together, the result will be $c + d$ times $a + b$.

16. Multiply $ax - 3ay + xy$ by $3ay + ax$.

$$\begin{array}{l}
ax - 3ay + xy \\
3ay + ax \\
\hline
3a^2xy - 9a^2y^2 + 3axy^2 \\
a^2x^2 - 3a^2xy + ax^2y \\
\hline
a^2x^2 - 9a^2y^2 + 3ayx^2 + ax^2y.
\end{array}$$

In adding these two products, the quantity $3\,a^2xy$ occurs twice, with different signs; they therefore destroy each other and do not appear in the result.

17. Multiply $5ad + 3acd - 5a^2c$
by $2a^2c + 2ad$.

18. Multiply $13a^2ry - 2aby^2 + 3cy^3$
by $5cy^3 + 7aby^2 + 3$.

19. Multiply $11ac^2 + 3a^2c - 4a^3$
by $2a^2c + ac^2$

20. Multiply $a^2 - 2ac + c^2$ by $a + c$.

21. " $3a^4 - 3b^4$ by $2a^2 + 2b^2$

22. " $3b + 2c$ by $2a - 3b$.

$$\begin{array}{l}
3b + 2c \\
2a - 3b \\
\hline
6ab + 4ac \\
-9b^2 - 6bc \\
\hline
6ab + 4ac - 9b^2 - 6bc.
\end{array}$$

If $3b + 2c$ be multiplied by $2a$ only, the product will be too large by $3b$ times $3b + 2c$; hence this quantity must be multiplied by $3b$, and the product subtracted from $6ab + 4ac$.

This result may be proved by multiplying the multiplier by the multiplicand, for the product must be the same in both cases.

23. Multiply $2ad + 3bc + 2$ by $4ab - 2c$.

24. Multiply $6a^2b + 2ab^2$ by $2a^2b - b^2 - 1$.

25. " $19 - 5$ by $9 - 4$.

$$\begin{array}{rl}
19 - 5 & = 14 \\
9 - 4 & = 5 \\
\hline
171 - 45 & 70. \\
-76 + 20 & \\
\hline
\end{array}$$

$$171 - 45 - 76 + 20 = 191 - 121 = 70.$$

26. Multiply $a - b$ by $c - d$.

$$\begin{array}{r}
a - b \\
c - d \\
\hline
ac - bc \\
-ad + bd \\
\hline
ac - bc - ad + bd.
\end{array}$$

This operation is sufficiently manifest in the figures. In the letters, I first multiply $a - b$ by c, which gives $ac - bc$; but the multiplier is not so large as c by the quantity d, therefore the product $ac - bc$ is too large by d times $a - b$; this then must be multiplied by d and the product subtracted. $a - b$ multiplied by d gives $ad - bd$; and this subtracted from $ac - bc$ gives $ac - bc - ad + bd$. Hence it appears that if two terms having the sign — be multiplied together, the product must have the sign +.

From the preceding examples and observations, we derive the following general rule for multiplying compound quantities.

1. *Multiply all the terms of the multiplicand by each term of the multiplier, observing the same rules for the coefficients and letters as in simple quantities.*

2. With respect to the signs observe,

1st, *That if both the terms which are multiplied together, have the sign +, the sign of the product must be +.*

2d, *If one term be affected with +, and the other with —, the product must have the sign —.*

3d, *If both terms be affected with the sign —, the product must have the sign +.*

Or in more general terms, *If both terms have the same sign, whether + or —, the product must have the sign +, and if they have different signs, the product must have the sign —.*

27. Multiply $3a^2b - 2ac + 5$
by $7ab - 2ac - 1$.

$$\begin{array}{r}
21a^3b^2 - 14a^3bc + 35ab \\
-6a^3bc + 4a^2c^2 - 10ac \\
-3a^2b + 2ac - 5. \\
\hline
\end{array}$$

Product
$21a^3b^2 - 14a^3bc + 35ab - 6a^3bc + 4a^2c^2 - 8ac - 3a^2b - 5$.

28. Multiply $7m + 5n$ by $4m - 3n$.
29. " $a^2 + ay - y^2$ by $a - y$.
30. " $n^2 + nx + x^2$ by $n - x$.
31. " $a^2 + ab + b^2$ by $a^2 - ab + b^2$.
32. " $2x^2 - 3xy + 4y^2$ by $5x - 6xy - 2y^2$.
33. " $3a^2c - 5ac^2 + 2c^3$ by $2a^3c - 4a^2c^2 - 7ac^2$.
34. " $2a^3 - a^2x + 2$ by $3a - x - 3$.
35. " $7a^2b + 2b^2 - 1$ by $3a^2 - 2b^2 - 1$.

It is generally much easier to trace the effect produced by each of several quantities in forming the result, when the operations are performed upon letters, than when performed upon figures. The following are remarkable instances of this. They ought to be remembered by the learner, as frequent use is made of them in all analytical operations.

Let a and b represent any two numbers; $a + b$ will be their sum and $a - b$ their difference.

Multiply $a + b$ by $a - b$.

$$\begin{array}{l}
a + b \\
a - b \\
\hline
a^2 + a\,b \\
\quad - a\,b - b^2 \\
\hline
a^2 - b^2.
\end{array}$$

That is, *if the sum and the difference of two numbers be multiplied together, the product will be the difference of the second powers of these two numbers.*

Particular Example.

Let $a = 12$ and $b = 7$.

$a + b = 19$, and $a - b = 5$,

$a^2 = 144$, $b^2 = 49$.

$(a + b) \times (a - b) = 19 \times 5 = 95$,

and $a^2 - b^2 = 144 - 49 = 95$.

Multiply $a + b$ by $a + b$.

$$\begin{array}{l}
a + b \\
a + b \\
\hline
a^2 + a\,b \\
\qquad a\,b + b^2 \\
\hline
a^2 + 2\,a\,b + b^2.
\end{array}$$

That is, *the product of the sum of two numbers, by itself, or the second power of the sum of two numbers, is equal to the sum of the second powers of the two numbers, added to twice the product of the two numbers.*

Multiply $a - b$ by $a - b$.

The answer is $a^2 - 2\,a\,b + b^2$, which is the same as the last, except the sign before $2\,a\,b$.

Multiply $a^2 + 2\,a\,b + b^2$ by $a + b$, that is, find the third power of $a + b$.

Ans. $a^3 + 3\,a^2\,b + 3\,a\,b^2 + b^3$.

This is expressed in words thus: *the third power of the first, plus three times the second power of the first into the second, plus three times the first into the second power of the second, plus the third power of the second.*

Multiply $a^2 - 2ab + b^2$ by $a - b$.

Ans. $a^3 - 3a^2b + 3ab^2 - b^3$.

Which is the same as the last, except the signs before the second and last terms.

Instances of the use of the above formulas will frequently occur in this treatise.

Division of Algebraic Quantities.

XIV. The division of algebraic quantities will be easily performed, if we bear in mind that it is the reverse of multiplication, and that the divisor and quotient multiplied together must reproduce the dividend.

The quotient of ab divided by a is b, for a and b multiplied together produce ab. So ab divided by b gives a for a quotient, for the same reason.

If $6abc$ be divided by $2a$, the quotient is $3bc$.
If by $2b$, the quotient is $3ac$.
If by $2c$, the quotient is $3ab$.
If by $3b$, the quotient is $2ac$.
If by $3ab$, the quotient is $2c$.
If by $6a$ the quotient is bc.

For in all these instances the quotient multiplied by the divisor, produces the dividend $6abc$.

Examples.

1. How many times is $2a$ contained in $6abc$?

Ans. $3bc$ times, because $3bc$ times $2a$ is $6abc$.

2. If $6abc$ be divided into $2a$ parts, what is one of the parts?

Ans. $3bc$; because $2a$ times $3bc$ is $6abc$.

Hence we derive the following

Rule. *Divide the coefficient of the dividend by the coefficient of the divisor, and strike out the letters of the divisor from the dividend.*

3. Divide $16\,a\,b\,c$ by 4.
4. " $12\,a\,b\,c$ by $3\,a$.
5. " $20\,a\,b\,c$ by $10\,b\,c$.
6. " $18\,a\,b\,c\,d$ by $6\,a\,d$.
7. " $23\,a\,b\,c$ by $a\,b$.
8. " $17\,a\,d$ by $a\,d$.
9. " $4\,a^3$ by a^2.

Observe that $4\,a^3$ is the same as $4\,a\,a\,a$ and a^2 is the same as $a\,a$; $4\,a\,a\,a$ divided by $a\,a$ gives $4\,a$ for the quotient.

It was observed in multiplication, that when the same letter enters into both multiplier and multiplicand, the multiplication is performed by adding the exponents, thus a^3 multiplied by a^2 is $a^{3+2} = a^5$. In similar cases, *division is performed by subtracting the exponent of the divisor from that of the dividend.* a^5 divided by a^2 is $a^{5-2} = a^3$.

10. Divide $6\,a^2\,b^3\,c$ by $3\,a\,b^2$.
Ans. $2\,a\,b\,c$.
11. " $35\,b^3\,d^2$ by $b\,d$.
12. " $16\,a^6\,c^2$ by $4\,a^3\,c^2$.
13. " $18\,x^2\,y^2$ by $6\,y^2$.
14. " $48\,a^4\,x^3\,m$ by $16\,a^2\,x\,m$.
15. " $72\,a\,r^5\,m^3$ by $12\,a\,r^2$.
16. " $60\,p^2\,y^4$ by 60.
17. " $73\,a\,p^2$ by $a\,p^2$.
18. " $120\,a\,r^2\,t^6$ by $r\,t^2$.

The division of some compound quantities is as easy as that of simple quantities.

If $a + b + c$ be multiplied by d the product is

$$d\,(a + b + c) \text{ or } a\,d + b\,d + c\,d.$$

Therefore if $a\,d + b\,d + c\,d$ be divided by d, the quotient is $a + b + c$.

If $ad + bd + cd$ be divided by $a + b + c$, the quotient is d.

When a compound quantity is to be divided, let the quantity, if possible, be so arranged that the divisor may appear as one of the factors, and then that factor being struck out, the other factor will be the quotient.

19. Divide $12a^2b - 9ac$ by $3a$.

$$12a^2b - 9ac = 3a(4ab - 3c)$$

Ans. $4ab - 3c$.

Observe that a is a factor of both terms, and also 3. Hence the quantity $12a^2b - 9ac$, can be resolved into factors; thus $3(4a^2b - 3ac)$, or $a(12ab - 9c)$, or $3a(4ab - 3c)$. In the last form the divisor $3a$ appears as one factor, and the other factor $4ab - 3c$ is the quotient.

Note. Any simple quantity, which is a factor of all the terms of any compound quantity, is a factor of the whole quantity; and that factor being taken out of all the terms, the terms as they then stand, taken together, will form the other factor.

20. Divide $8a^2b^3 - 16a^3b^2c$ by $2ab - 4a^2c$.

$$8a^2b^3 - 16a^3b^2c = 4ab^2(2ab - 4a^2c.)$$

Ans. $4ab^2$.

21. Divide $3abc - 15ab^2d + 9a^3bd^3$ by $3ab$.

22. Divide $15a^3bc - 30a^5c^2 + 25a^3cd$ by $5a^2c$.

23. Divide $36a^{13}b^2c - 28a^{11}b^4c^2 + 40a^9b^6c^3$ by $9a^6 - 7a^4b^2c + 10a^2b^4c^2$.

24. Divide $42a^7 - 84a^{10}b^2c$ by $1 - 2a^3b^2c$.

Algebraic Fractions.

XV. When the dividend does not contain the same letters as the divisor, or but part of those of the divisor, the division cannot be performed in this way. It can then only be expressed. The usual way of expressing division, as has already been explained, is by writing the divisor under the dividend in the form of a fraction. Thus a divided by b is expressed $\frac{a}{b}$.

This gives rise to fractions in the same manner as in arithmetic. It was shown in arithmetic, that a fraction properly expresses a quotient. Algebraic fractions are subject to precisely the same rules as fractions in arithmetic. Many of the operations are more easily performed on algebraic fractions.

In these, as in arithmetic, it must be kept in mind, that the denominator shows into how many parts a unit is divided; and the numerator shows how many of those parts are used; or the denominator shows into how many parts the numerator is divided.

I shall here briefly recapitulate the rules for the operations on fractions, referring the learner to the Arithmetic for a more full developement of their principles.

$$2 \text{ times } \frac{3}{11} = \frac{6}{11}.$$

$$2 \text{ times } \frac{a}{b} = \frac{2a}{b}.$$

$$c \text{ times } \frac{a}{b} = \frac{ac}{b}.$$

$\frac{3}{5}$ of 7 is $\frac{21}{5}$; for $\frac{1}{5}$ of 7 is $\frac{7}{5}$, and $\frac{3}{5}$ is 3 times as much. $\frac{2}{7}$ of a is $\frac{2a}{7}$; for $\frac{1}{7}$ of a is $\frac{a}{7}$, and $\frac{2}{7}$ is 2 times as much. The $\frac{a}{b}$ part of c is $\frac{ac}{b}$; for $\frac{1}{b}$ of c is $\frac{c}{b}$, and $\frac{a}{b}$ is a times as much.

Hence, *to multiply a fraction by a whole number, or a whole number by a fraction, multiply the numerator of the fraction and the whole number together, and divide by the denominator.*

Arith. Articles XV. & XVI.

Examples.

1. Multiply $\frac{a+b}{c}$ by 2. *Ans.* $\frac{2a+2b}{c}$.

2. Multiply $\frac{3a+2b^2}{ac}$ by bd.

Ans. $\frac{3abd+2b^3d}{ac}$.

3. Multiply $\frac{3bc-2a}{5a-13c}$ by $4b^2$.

Ans. $\frac{12b^3c-8ab^2}{5a-13c}$.

4. Multiply $\frac{2ab-bc}{3ab}$ by $5ac+3c^2$.

5. Multiply $\frac{5ac-2m^2}{13ac}$ by $5ab-3n$.

6. Multiply $16ax^2-3bx$ by $\frac{2m-3x}{2a+7x^2}$.

Division of Fractions.

XVI. 1. Divide $\frac{4a}{7}$ by 2, or find $\frac{1}{2}$ of $\frac{4a}{7}$.

Ans. $\frac{2a}{7}$.

2. Divide $\frac{ab}{c}$ by a, or find $\frac{1}{a}$ of $\frac{ab}{c}$.

Ans. $\frac{b}{c}$.

3. Divide $\frac{6a^2b}{cd}$ by $3a$, or find $\frac{1}{3a}$ of $\frac{6a^2b}{cd}$.

Ans. $\frac{2ab}{cd}$.

4. Divide $\frac{a}{b}$ by 2, or find $\frac{1}{2}$ of $\frac{a}{b}$.

This cannot be done like the others, but it may be done by multiplying the denominator as in Arith. Art. XVII. For the fraction $\frac{a}{b}$ denotes, that one is divided into as many equal parts as there are units in b, and that as many of these parts are used as there are units in a; or that a is divided into as many equal parts as there are units in b; hence if it be divided into twice as many parts, the parts will be only one half as large, and the fraction will have only one half the value.

Hence $\frac{a}{b}$ divided by 2, is $\frac{a}{2b}$.

So $\frac{b}{c}$ divided by d, is $\frac{b}{cd}$.

5. Divide $\frac{3ab}{2cd}$ by $4d$. *Ans.* $\frac{3ab}{8cd^2}$.

Hence, *to divide a fraction by a whole number, divide the numerator; or when that cannot be done, multiply the denominator by the divisor.*

6. Divide $\frac{6ab}{d}$ by $3a$.

7. Divide $\frac{14a^2b^3}{cd}$ by $7ab$.

8. Divide $\frac{2a^6cb}{3dm}$ by $2a^3c$.

9. Divide $\frac{3b^2}{5ac}$ by b^2.

10. Divide $\frac{7bc}{a}$ by 3.

11. Divide $\frac{4ac}{bd}$ by 5.

12. Divide $\frac{17acd}{mnr}$ by $2bm$.

13. Divide $\frac{13ab}{5cd^2}$ by $3c^2d$.

14. Divide $\frac{27mr}{7ac^3d}$ by $8a^2cb$.

15. Divide $\frac{3a-2b}{2bc}$ by $3ad$.

16. Divide $\frac{7am-13bc}{2ad-5b^2}$ by $3ab$.

17. Divide $\frac{12acd}{5an-mn^2}$ by $7bn^2$.

18. Divide $\frac{17c - 3x^2}{2a^2n - 7n^2}$ by $4a^2 + 3n$.

19. Divide $\frac{37ad}{46 + 3x}$ by $4b - 3x$.

20. Divide $\frac{2a - d}{3a - 4cd + 1}$ by $7a + 4cd - 1$.

21. What is $\frac{2}{3}$ of $\frac{a}{b}$? $\frac{1}{3}$ of $\frac{a}{b}$ is $\frac{a}{3b}$ and $\frac{2}{3}$ is twice as much, that is, $\frac{2a}{3b}$.

22. What is the $\frac{a}{b}$ part of $\frac{c}{d}$? $\frac{1}{b}$ of $\frac{c}{d}$ is $\frac{c}{bd}$, and $\frac{a}{b}$ is a times as much, that is, $\frac{ac}{bd}$.

That is, $\frac{c}{d} \times \frac{a}{b} = \frac{ac}{bd}$.

Hence, *to multiply one fraction by another, multiply the numerators together for a new numerator, and the denominators together for a new denominator.* *Arith. Art.* XVII.

23. Multiply $\frac{2a}{3c}$ by $\frac{b}{2m}$. *Ans.* $\frac{2ab}{6cm}$.

24. Multiply $\frac{3ad}{4bc}$ by $\frac{3am}{5c^2n}$.

25. Multiply $\frac{12anx}{13bry}$ by $\frac{3ax^2}{2by}$.

26. What is $\frac{2a^2m}{5cd}$ of $\frac{8bf}{3md}$?

27. What is $\frac{4b^2d}{9ax^2}$ of $\frac{7bm}{13nx}$?

28. Multiply $\frac{2a}{3b + c}$ by $\frac{2ac - bc}{5ab}$.

29. Multiply $\frac{2am^2 - 3a^2m}{4ac + 2c}$ by $\frac{5am^2}{2am - 5c}$.

30. Multiply $\frac{2ac+3}{5bc-2d^3}$ by $\frac{3ab-c}{5ac-2ad}$.

31. Multiply $\frac{2a-m+3m^2}{7am^3}$ by $\frac{13ac}{17am^2-c+5}$.

We have seen that a fraction may be divided by multiplying its denominator, because the parts are made smaller; on the contrary, a fraction may be multiplied by dividing its denominator, because the parts are made larger. Arith. Art. XVIII. If the denominator be divided by 2, the unit is divided into only one half as many parts; consequently the parts must be twice as large as before. If the denominator be divided by 5, the unit is divided into only one fifth as many parts; hence the parts must be five times as large as before, and if the same number of parts be used as at first, the value of the fraction will be five times as great and so on.

32. Multiply $\frac{3a}{20}$ by 5. *Ans.* $\frac{3a}{4}$.

33. Multiply $\frac{a}{bc}$ by b.

If we divide the denominator by b, the fraction becomes $\frac{a}{c}$, in which a is divided into $\frac{1}{b}$ part as many parts; hence the parts, and consequently the fraction is b times as large as before.

34. Multiply $\frac{3a}{6bc}$ by $2c$.

35. Multiply $\frac{17ab}{32c^3d}$ by $8c^3d$.

36. Multiply $\frac{16}{42a^2m^2}$ by $7am^2$.

37. Multiply $\frac{ab}{25m^5x}$ by $5mx$.

38. Multiply $\frac{3}{5a}$ by 5.

39. Multiply $\frac{7}{8ab}$ by ab.

40. Multiply $\frac{3ac}{4ab^2 - 4bc}$ by $4b$.

41. Multiply $\frac{17 - 4bc}{16a^2 - 12a^3b - 4a^2}$ by $4a^2$.

42. Multiply $\frac{23m - 13}{35m^2cd - 7m^6c + 42m^4ac^3}$ by $7m^2c$.

43. Multiply $\frac{3}{5}$ by 5.

Dividing the denominator by 5 it becomes $\frac{3}{1}$, or 3.

Multiply $\frac{a}{b}$ by b.

Dividing the denominator by b it becomes $\frac{a}{1}$, or a.

44. Multiply $\frac{3ac}{5bd}$ by $5bd$. *Ans.* $\frac{3ac}{1} = 3ac$.

In fact $\frac{1}{b}$ multiplied by b is $\frac{b}{b} = 1$, and $\frac{a}{b}$ being a times as much as $\frac{1}{b}$, must give a product a times as large, or a times 1, which is a.

Hence, *if a fraction be multiplied by its denominator, the product will be the numerator.*

45. Multiply $\frac{7acm}{5bd}$ by $5bd$.

46. Multiply $\frac{25}{3bc}$ by $3bc$.

47. Multiply $\frac{18x}{4bm^2}$ by $4bm^2$.

48. Multiply $\frac{12m^2y^3}{bdn^2x}$ by bdn^2x.

49. Multiply $\frac{13ab - m}{17a^2}$ by $17a^2$.

50. Multiply $\frac{15ac + 37bc}{10ab - 2c}$ by $10ab - 2c$.

51. Multiply $\frac{47am^2 + 3b - c}{ax^2 - 3a^2m + b}$ by $ax^2 - 3a^2m + b$.

Two ways have been shown to multiply fractions, and two ways to divide them.

To multiply a fraction,	*multiply*	*the numerator*
To divide a fraction,		*the denominator.*
To divide a fraction,	*divide*	*the numerator.*
To multiply a fraction,		*the denominator.*

Arith. Art. XVIII.

Reducing Fractions to Lower Terms.

XVII. *If both numerator and denominator be multiplied by the same number, the value of the fraction will not be altered.*

Arith. Art. XIX.

For multiplying the numerator multiplies the fraction, and multiplying the denominator divides it; hence it will be multiplied and the product divided by the multiplier, which reproduces the multiplicand.

In other words, $\frac{a}{b}$ signifies that a contains b a certain number of times, if a is as large or larger than b; or a part of one time, if b is larger than a. Now it is evident that $2a$ will contain $2b$ just as often, since both numbers are twice as large as before.

So dividing both numerator and denominator, both divides and multiplies by the same number.

$$\frac{3}{5} = \frac{2 \times 3}{2 \times 5} = \frac{6}{10} = \frac{7 \times 3}{7 \times 5} = \frac{21}{35} = \frac{3 \times b}{5 \times b} = \frac{3b}{5b}.$$

$$\frac{a}{b} = \frac{2a}{2b} = \frac{5a}{5b} = \frac{ac}{bc} = \frac{2acd}{2bcd}.$$

$$\frac{6ab}{9bc} = \frac{3b \times 2a}{3b \times 3c} = \frac{2a}{3c}.$$

Hence, if a fraction contain the same factor both in the numerator and denominator, it may be rejected in both, that is, both may be divided by it. This is called reducing fractions to lower terms.

1. Reduce $\frac{9\,a\,m}{15\,b\,c\,m}$ to its lowest terms. *Ans.* $\frac{3\,a}{5\,b\,c}$.

2. Reduce $\frac{12\,a\,x^2}{16\,a^2\,x^3}$ to its lowest terms. *Ans.* $\frac{3}{4\,a\,x}$.

3. Reduce $\frac{5\,a\,m^2}{30\,b\,m}$ to its lowest terms. *Ans.* $\frac{a\,m}{6\,b}$.

4. Reduce $\frac{32\,b\,m^2\,y^5}{16\,b^2\,r\,y^3}$ to its lowest terms.

5. Reduce $\frac{52\,a\,b^5\,x^3}{13\,a^3\,b^2\,x^2}$ to its lowest terms.

6. Reduce $\frac{15\,a^3\,c^2 - 25\,a^7}{5\,a^2\,b\,c + 55\,a^6\,b}$ to its lowest terms.

7. Reduce $\frac{27\,m^5\,x - 54\,x^3}{108\,a\,x^2 + 81\,x - 90\,m^2\,x^3}$ to its lowest terms.

8. Divide $35\,a^2\,b\,m^8\,x^3$ by $7\,a^3\,n\,m^5\,x$.

Write the divisor under the dividend in the form of a fraction, and reduce it to its lowest terms.

Ans. $\frac{5\,b\,m^3\,x^2}{a\,n}$.

9. Divide $27\,b^2\,m\,y^3$ by $21\,b^2\,m^2\,y$.

Ans. $\frac{9\,y^2}{7\,m}$.

10. Divide $56\,b\,r^2\,y$ by $7\,b^3\,n\,y^2$.

11. Divide $54\,m^2\,n\,r^3\,y$ by $36\,b\,m\,y^3$.

12. Divide $18\,c^3\,d\,m\,x^6$ by $63\,c\,m^6\,r\,x^3$.

13. Divide $115\,r\,s\,y^3$ by $15\,r\,s\,y$.

14. Divide $128\,a^4\,c^5\,r\,x^2$ by $48\,a^2\,m\,r^7\,x^2$.

15. Divide $17 a c x$ by $13 a c^3 x^4$.

16. Divide $28 a^2 c y$ by $14 a^3 y^2$.

17. Divide $36 a^2 m^3 y$ by $54 a^5 m y^3$.

18. Divide $75 a^7 b y^2$ by $35 a^2 c^2 y^5 x$.

19. Divide $a + b$ by $2 c - d$.

20. Divide $2 a^2 c - 7 a^6 b c + 15 a^3 c d$ by $13 a^7 c d$.

21. Divide $18 a^3 m^3 - 54 a^5 m^2 + 42 a^2 m^4$ by $30 a^6 m^2 d - 12 a^2 c m^3$.

22. Divide $(a + b)(13 a c + b c)$ by $(m^2 - c)(a + b)$.

23. Divide $3 c^2 (a - 2 c)^3$ by $2 b c^3 (a - 2 c)^2$.

24. Divide $36 b^3 c^2 (2 a + d)^2 (7 b - d)^5$ by $12 b^6 (2 a + d)^2 (7 b - d)^5 (a - d)$.

Addition and Subtraction of Fractions.

XVIII. Add together $\frac{a}{b}$ and $\frac{c}{d}$ and $\frac{e}{f}$.

This addition may be expressed by writing the fractions one after the other with the sign of addition between them; thus $\frac{a}{b} + \frac{c}{d} + \frac{e}{f}$.

N. B. When fractions are connected by the signs + and —, the sign should stand directly in a line with the line of the fraction.

It is frequently necessary to add the numerators together, in which case, the fractions, if they are not of the same denomination, must first be reduced to a common denominator, as in Arithmetic, Art. XIX.

1. Add together $\frac{3}{7}$ and $\frac{2}{7}$. *Ans.* $\frac{3+2}{7} = \frac{5}{7}$.

2. Add together $\frac{a}{b}$ and $\frac{c}{b}$. *Ans.* $\frac{a+c}{b}$.

3. Add together $\frac{3a}{cd}$ and $\frac{2a}{cd}$. *Ans.* $\frac{3a+2a}{cd}=\frac{5a}{cd}$.

4. Add together $\frac{2a}{3cd}$ and $\frac{5ab}{3cd}$. *Ans.* $\frac{2a+5ab}{3cd}$.

5. Add together $\frac{3}{5}$ and $\frac{2}{7}$.

These must be reduced to a common denominator. It has been shown above that if both numerator and denominator be multiplied by the same number, the value of the fraction will not be altered. If both the numerator and denominator of the first fraction be multiplied by 7, and those of the second by 5, the fractions become $\frac{21}{35}$ and $\frac{10}{35}$. They are now both of the same denomination, and their numerators may be added. The answer is $\frac{31}{35}$.

6. Add together $\frac{a}{b}$ and $\frac{c}{d}$

Multiply both terms of the first by d, and of the second by b, they become $\frac{ad}{bd}$ and $\frac{bc}{bd}$. The denominators are now alike and the numerators may be added.

The answer is $\frac{ad+bc}{bd}$.

7. Add together $\frac{a}{b}$, $\frac{c}{d}$, $\frac{e}{f}$, and $\frac{g}{h}$.

In all cases the denominators will be alike if both terms of each fraction be multiplied by the denominators of all the others. For then they will all consist of the same factors.

Applying this rule to the above example, the fractions become $\frac{adfh}{bdfh}$, $\frac{bcfh}{bdfh}$, $\frac{bdeh}{bdfh}$ and $\frac{bdfg}{bdfh}$.

The answer is $\frac{adfh+bcfh+bdeh+bdfg}{bdfh}$.

8. Add together $\frac{3a}{2bc}$ and $\frac{2c}{5d}$. *Ans.* $\frac{15ad+4bc^2}{10bcd}$.

It was shown in Arithmetic, Art. XXII, that a common denominator may frequently be found much smaller than that produced by the above rule. This is much more easily done in algebra than in arithmetic.

9. Add together $\frac{a}{bc^2}$, $\frac{d}{be}$, and $\frac{f}{cg}$.

Here the denominators will be alike, if each be multiplied by all the factors in the others not common to itself. If the first be multiplied by eg, the second by c^2g, and the third by bce, each becomes bc^2eg. Then each numerator must be multiplied by the same quantity by which its denominator was multiplied, that the value of the fractions may not be altered. The fractions then become $\frac{aeg}{bc^2eg}$, $\frac{c^2dg}{bc^2eg}$, and $\frac{ebcf}{cb^2eg}$.

The answer is $\frac{aeg + c^2dg + bcef}{bc^2eg}$.

10. Add together $\frac{2ac}{be}$ and $\frac{3bf}{2dg}$.

11. Add together $\frac{5am}{2r}$, $\frac{ec}{3b}$, and $\frac{en}{h}$.

12. Add together $\frac{3a}{2mn}$ and $\frac{3b}{3mp}$.

13. Add together $\frac{ar}{5bm^2}$, $\frac{2c}{5bn}$, and $\frac{3cd}{2m^3n}$.

14. Add together $\frac{3m^2s}{n^5r^3}$, and $\frac{2ar}{3mn^3r}$.

15. Add together $\frac{15a}{4}$, $\frac{bc}{2h^3n^2}$, and $\frac{2m^3r}{h^2n^3x}$.

16. Add together $\frac{2ac}{3b}$, and $13cd$.

17. Add together $\frac{2am}{4an}$, and $2ac - 5b$.

18. Add together $\frac{13\,a^2\,n^2 - 4\,c^2}{2\,a^2\,n}$ and $11\,a\,c - 5\,n$.

19. Add together $\frac{15\,a^3\,m\,c - 2\,ab}{7\,a\,m^2\,b}$ and $\frac{2\,a^2\,c}{3\,b\,m}$.

20. Add together $\frac{13\,a^2\,b - 2\,c}{4\,a\,b}$, and $\frac{7\,a\,b + 8\,c}{2\,b + 16\,a\,b}$.

21. Subtract $\frac{e}{b\,c^2}$ from $\frac{3\,a}{2\,b\,c}$.

This subtraction may be expressed thus,

$$\frac{3\,a}{2\,b\,c} - \frac{e}{b\,c^2}.$$

But if they are reduced to a common denominator, the numerators may be subtracted.

Ans. $\frac{3\,a\,c - 2\,e}{2\,b\,c^2}$.

22. Subtract $\frac{2\,a\,b}{3\,c^2\,d}$ from $\frac{5\,m\,n}{2\,c^2\,x}$.

23. Subtract $\frac{3\,c\,d}{7\,b^3\,m^2\,x}$ from $\frac{c\,d\,e}{21\,b\,m^3\,x^2}$.

24. Subtract $\frac{b\,d}{n\,x^3\,y^2}$ from $\frac{3\,r\,x}{2\,m\,y^5}$.

25. Subtract $\frac{5\,x\,y}{2\,m^5\,s^2\,t}$ from $\frac{2\,r\,y}{3\,m^2\,s\,t^3}$.

26. Subtract $\frac{17\,r\,x^3}{3\,m\,b}$ from $\frac{15\,x\,y}{5\,m^2\,b^3}$.

27. Subtract $\frac{13\,m^3\,y}{7\,n^5\,x}$ from $\frac{11\,r\,s}{3\,n^2\,x^3}$.

28. From $13\,a\,c + b\,c$ subtract $\frac{3\,a\,c}{2\,b\,m}$.

29. From $\frac{2\,a^2\,x^2 - 14\,x}{2\,a^2\,m\,x}$ subtract $\frac{13\,a\,x}{14\,a\,m}$.

30. From $\frac{27\,a\,d}{2\,b\,c^2}$ subtract $\frac{2\,a\,b\,d - 3\,m^2\,c}{4\,b^2\,c^2}$.

Solution.

$$\frac{27\,a\,d}{2\,b\,c^2} - \frac{2\,a\,b\,d - 3\,c\,m^2}{4\,b^2\,c^2} = \frac{(27\,a\,d)\,2\,b}{(2\,b\,c^2)\,2\,b}$$

$$- \frac{2\,a\,b\,d - 3\,c\,m^2}{4\,b^2\,c^2} = \frac{54\,a\,b\,d}{4\,b^2\,c^2}$$

$$- \frac{2\,a\,b\,d - 3\,c\,m^2}{4\,b^2\,c^2} = \frac{54\,a\,b\,d - 2\,a\,b\,d + 3\,c\,m^2}{4\,b^2\,c^2}$$

$$= \frac{52\,a\,b\,d + 3\,c\,m^2}{4\,b^2\,c^2}$$

which is the answer.

When the fraction $\frac{2\,a\,b\,d - 3\,c\,m^2}{4\,b^2\,c^2}$ was subtracted, the sign — was changed to +. See Art. VI, example 6th.

31. From $\frac{5\,n\,x^2 - 10\,a\,d\,x}{12\,a\,d}$

Subtract $\frac{13\,n\,x^2 - 5\,m\,x^2 + 17}{6\,m\,x}$.

32. From $\frac{12\,a\,n\,x}{3\,d\,x^2 - 5}$ subtract $\frac{11\,a\,n}{4\,d\,x}$.

XIX. *Division of whole numbers by Fractions, and Fractions by Fractions.*

1. How many times is $\frac{3}{5}$ contained in 7?

Ans. $\frac{1}{5}$ is contained in 7, 35 times, and $\frac{3}{5}$ is contained $\frac{1}{3}$ as many times; that is, $\frac{35}{3}$ or $11\frac{2}{3}$ times.

2. How many times is $\frac{5}{8}$ contained in a?

Ans. $\frac{1}{8}$ is contained in a, $8\,a$ times, and $\frac{5}{8}$ is contained $\frac{1}{5}$ as many times; that is, $\frac{8\,a}{5}$.

3. How many times is $\frac{a}{b}$ contained in c?

Ans. $\frac{1}{b}$ is contained $b\,c$ times in c, and $\frac{a}{b}$ is contained $\frac{1}{a}$ as many times; that is, $\frac{b\,c}{a}$.

Arith. Art. XXIII.

4. Of what number is c the $\frac{a}{b}$ part?

Ans. If c is the $\frac{a}{b}$ part of some number, $\frac{c}{a}$ will be $\frac{1}{b}$ part of the same number, and $\frac{c}{a}$ is $\frac{1}{b}$ part of $\frac{b\,c}{a}$.

Arith. Art. XXIV.

Hence, *to divide a whole number by a fraction, multiply it by the denominator of the fraction, and divide the product by the numerator.*

How many times is $\frac{3}{5}$ contained in $\frac{7}{8}$.

Solution. Reducing them to a common denominator, $\frac{3}{5}$ is $\frac{24}{40}$, and $\frac{7}{8}$ is $\frac{35}{40}$. $\frac{24}{40}$ is contained in $\frac{35}{40}$ as many times as 24 is contained in 35; that is, $\frac{35}{24}$ or $1\frac{11}{24}$. *Ans.* $1\frac{11}{24}$.

6. How many times is $\frac{a}{b}$ contained in $\frac{c}{d}$?

Solution. Reducing them to a common denominator, $\frac{a}{b}$ is $\frac{a\,d}{b\,d}$, and $\frac{c}{d}$ is $\frac{b\,c}{b\,d}$. $\frac{a\,d}{b\,d}$ is contained in $\frac{b\,c}{b\,d}$ as many times as $a\,d$ is contained in $b\,c$; that is, $\frac{b\,c}{a\,d}$. *Ans.* $\frac{b\,c}{a\,d}$.

7. Of what number is $\frac{c}{d}$ the $\frac{a}{b}$ part?

Solution. If $\frac{c}{d}$ is the $\frac{a}{b}$ part of some number, $\frac{1}{a}$ part of $\frac{c}{d}$ is $\frac{1}{b}$ part of that number; $\frac{1}{a}$ part of $\frac{c}{d}$ is $\frac{c}{a\,d}$ and $\frac{c}{a\,d}$ is $\frac{1}{b}$ part of $\frac{b\,c}{a\,d}$. *Ans.* $\frac{b\,c}{a\,d}$.

Hence, *to divide a fraction by a fraction, multiply the numerator of the dividend by the denominator of the divisor, and the denominator of the dividend by the numerator of the divisor.*

Or more generally, *when the divisor is a fraction, multiply the dividend (whether whole number or fraction) by the divisor inverted.*
Arith. Arts. XXIII. and XXIV.

8. Divide $3\,a\,b$ by $\frac{2}{3}$.

9. Divide $13\,a$ by $\frac{b}{c}$.

10. Divide $17\,a\,m$ by $\frac{2\,c}{b}$.

11. Divide $a\,c\,t$ by $\frac{3\,b\,c}{2\,a}$.

12. Divide $3\,a\,x$ by $\frac{2\,a^2\,m}{3\,x\,y}$.

13. Divide $2\,a\,c - b\,c$ by $\frac{3\,a}{5\,c}$.

14. Divide $17\,a\,x^2 - 2\,b\,x + c\,x$ by $\frac{13\,a\,b\,x - 2\,x}{7\,a^2\,c}$.

15. Divide $11\,a\,x^3 - 3\,x$ by $\frac{2\,x}{7\,a\,c\,x - 3\,a\,c}$.

16. Divide $\frac{b\,c}{d}$ by $\frac{3\,a\,c}{m}$.

17. Divide $\frac{2\,c\,d}{3\,a\,y^2}$ by $\frac{2\,x\,y}{5\,a\,d^2}$.

18. Divide $\frac{17\,a^3\,m}{5\,x^5\,y^3}$ by $\frac{3\,a^5\,n^2}{7\,x^3\,y^7}$.

19. Divide $\frac{36\,a^7\,n}{35\,b^3\,m\,y^2}$ by $\frac{45\,a^2\,n^5\,s}{21\,b^5\,m^2\,y}$.

20. Divide $\frac{13\,a + 2\,b\,a}{12\,a\,x}$ by $\frac{13\,a\,b - 2\,a\,x^2 + 7}{9\,a\,x\,c\,x}$.

21. Divide $\frac{2\,a - 3\,c\,d}{2\,a\,m + 5\,a\,x}$ by $\frac{2\,a\,m - 5\,a\,x}{2\,a + 3\,c\,d}$.

22. Divide $\frac{5\,m\,x - 2\,d}{3\,m\,y + 3\,m\,d}$ by $\frac{13\,a\,y}{5\,d - m\,x}$.

Division of Compound Quantities.

XX. Sometimes division may actually be performed when both divisor and dividend are compound quantities. Since division is the reverse of multiplication, the proper method to discover how to perform it, is to observe how a product is formed by multiplication.

Multiply $2\,a^3\,b - 3\,a^2\,b^2\,c + a\,b^3\,c^2$
by $4\,a^2\,b^2 + 2\,a\,b\,c.$

$$8\,a^5\,b^3 - 12\,a^4\,b^4\,c + 4\,a^3\,b^5\,c^2 + 4\,a^4\,b^2\,c - 6\,a^3\,b^3\,c^2 + 2\,a^2\,b^4\,c^3.$$

Observe that each term of the multiplier is multiplied separately into each term of the multiplicand. The product therefore must consist of a number of terms equal to the product of the number of terms in the multiplicand by the number of terms in the multiplier. If the product be divided by the multiplicand, the multiplier must be reproduced, and if by the multiplier, the multiplicand must be reproduced.

The three terms $8\,a^5\,b^3 - 12\,a^4\,b^4\,c + 4\,a^3\,b^5\,c^2$ of the product were produced by multiplying the three terms of the multiplicand by the first term of the multiplier, $4\,a^2\,b^2$. Therefore, if these three terms be divided by $4\,a^2\,b^2$, the quotient will be the multiplicand.

Again, the three terms

$$4\,a^4\,b^2\,c - 6\,a^3\,b^3\,c^2 + 2\,a^2\,b^4\,c^3$$

of the product were formed by multiplying each term of the multiplicand by $2\,a\,b\,c$. Therefore, if these three terms be divided by $2\,a\,b\,c$, the quotient will be the multiplicand.

Hence we see that the whole division might be performed by any one term of the divisor, if all the terms of the dividend which depend on that term and the quotient could be ascertained. This cannot often be done by inspection; for in many products, though at first there are as many terms as there are in the multiplicand and multiplier together, some of the terms are united together by addition or subtraction, and some disappear entirely. Even if all the terms did remain entire, they could not be easily distinguished.

However, one term may always be distinguished, and from it one term of the quotient may be obtained.

Divide $4a^4 - 9a^2b^2 + 6ab^3 - b^4$
by $2a^2 - 3ab + b^2$.

First, it is evident that the highest power of either letter in the dividend, must have been produced by multiplying the highest power of that letter in the divisor by the highest power of the same letter in the quotient; for in order to produce the dividend, each term of the divisor must be multiplied by every term of the quotient. Therefore, if $4a^4$ be divided by $2a^2$ it must give a term of the quotient. Or, if $-b^4$ be divided by b^2 it must give a term of the quotient. Let the quantities be arranged according to the powers of the letter a.

$$\begin{array}{ll}
\text{Dividend.} & \text{Divisor.} \\
4a^4 - 9a^2b^2 + 6ab^3 - b^4 & \big(\, 2a^2 - 3ab + b^2 \\
4a^4 - 6a^3b + 2a^2b^2 & \big(\, \overline{2a^2 + 3ab - b^2} \ \text{quotient.} \\
\hline
\quad 6a^3b - 11a^2b^2 + 6ab^3 - b^4 & \\
\quad 6a^3b - 9a^2b^2 + 3ab^3 & \\
\hline
\qquad -2a^2b^2 + 3ab^3 - b^4 & \\
\qquad -2a^2b^2 + 3ab^3 - b^4 & \\
\hline
\end{array}$$

I divide $4a^4$ by $2a^2$, which gives $2a^2$ for the first term of the quotient. Now in forming the dividend, every term of the divisor was multiplied by this term of the quotient, therefore I multiply the divisor by this term, by which means I find all the terms of the dividend, which depend on this term. They are

$$4a^4 - 6a^3b + 2a^2b^2.$$

Here is a term $6a^3b$ which is not in the dividend, this must have disappeared in the product. The term $2a^2b^2$ is not found alone, but it is like $9a^2b^2$ and must have disappeared by uniting with some other term to form that. I subtract these three terms from the dividend, and there remains

$$6a^3b - 11a^2b^2 + 6ab^3 - b^4.$$

which does not depend at all on the term $2a^2$ of the quotient, but which was formed by multiplying each remaining term of the quotient by all the terms of the divisor. This then is a new dividend, and to find the next term of the quotient we must proceed exactly as before; that is, divide the term of the dividend containing the highest power of a, which is $6a^3b$, by $2a^2$ of the divisor, because this must have been formed by multiplying $2a^2$ by the highest remaining power of a in the quotient. This gives for the quotient $+3ab$. I multiply each term of the divisor by this, and subtract the product as before, and for the same reason. The remainder is

$$-2a^2b + 3ab^3 - b^4,$$

which depends only on the remaining part of the quotient. The highest power of a, viz. $2a^2b^2$, must have been produced by multiplying some term of the quotient by $2a^2$ of the divisor; therefore I divide by this again, and obtain $-b^2$ for the quotient. I multiply by this and subtract as before, and there is no remainder, which shows that the division is completed.

By the above process I have been enabled to discover all the terms of the dividend produced by multiplying the first term of the divisor by each term of the quotient. If both be arranged according to the powers of the letter b, and the same course pursued, the same quotient will be obtained, but in a reversed order.

In the division the term $-2a^2b^2$ has the sign $-$. Here we must observe that the divisor and quotient multiplied together must reproduce the dividend.

If $+ab$ be divided by $+a$, the quotient must be $+b$, because $+a \times +b$ gives $+ab$.

If $-ab$ be divided by $+a$, the quotient must be $-b$, because $+a \times -b$ gives $-ab$.

If $+ab$ be divided by $-a$, the quotient must be $-b$, because $-a \times -b$ gives $+ab$.

If $-ab$ be divided by $-a$, the quotient must be $+b$, because $-a \times +b$ gives $-ab$.

The rule for signs therefore is the same as in multiplication.

When the signs are alike, that is, both + or both —, the sign of the product must be +; but when the signs are unlike, that is, one + and the other —, the sign of the quotient must be —.

By the reasoning above we derive the following rule for division of compound numbers.

Arrange the dividend and divisor according to the powers of some letter. Divide the first term of the dividend by the first term of the divisor, and write the result in the quotient. Multiply all the terms of the divisor by the term of the quotient thus found, and subtract the product from the dividend. The remainder will be a new dividend, and in order to find the next term of the quotient, proceed exactly as before; and so on until there is no remainder.

Sometimes, however, there will be a remainder, such that the first term of the divisor, will not divide either term of it; in which case the division can be continued no farther, and the remainder must be written over the divisor in the form of a fraction, and annexed to the quotient as in arithmetic.

Divide $2a^5 - 11a^4b + 11a^3b^2 + 13a^2b^3$ by $2a - b$.

$$\begin{array}{l|l}
 & 2a - b \\ \cline{2-2}
 & a^4 - 5a^3b + 3a^2b^2 + 8ab^3 \\
2a^5 - 11a^4b + 11a^3b^2 + 13a^2b^3 & + 4b^4 + \dfrac{4b^5}{2a - b}. \\
2a^5 - a^4b & \\ \cline{1-1}
\quad - 10a^4b + 11a^3b^2 + 13a^2b^3 & \\
\quad - 10a^4b + 5a^3b^2 & \\ \cline{1-1}
\qquad\qquad 6a^3b^2 + 13a^2b^3 & \\
\qquad\qquad 6a^3b^2 - 3a^2b^3 & \\ \cline{1-1}
\qquad\qquad\qquad 16a^2b^3 & \\
\qquad\qquad\qquad 16a^2b^3 - 8ab^4 & \\ \cline{1-1}
\qquad\qquad\qquad\qquad 8ab^4 & \\
\qquad\qquad\qquad\qquad 8ab^4 - 4b^5 & \\ \cline{1-1}
\qquad\qquad\qquad\qquad\qquad 4b^5. &
\end{array}$$

In this example, the division may be continued until the remainder is $4\,b^6$, which cannot be divided by a, therefore it must be written over the divisor $2\,a - b$ as a fraction and added to the quotient.

Examples.

1. Divide $x^2 + 2\,a\,x + a^2$ by $a + x$.*
2. Divide $a^2 - b^2$ by $a + b$.
3. Divide $b^4 + 2\,b^2\,x + x^2$ by $b^2 + x$.
4. Divide $x^3 - y^3$ by $x^2 + x\,y + y^2$.
5. Divide $x^3 - y^3$ by $x + y$.
6. Divide $15\,a^2 + 2\,a\,b - 8\,b^2$ by $3\,a - 2\,b$.
7. Divide $x^3 - 2\,x\,y^2 + y^3$ by $x - y$.
8. Divide $a^3 - 9\,a^2 + 27$ by $9 - 6\,a + a^2$.
9. Divide $4\,a^4 - 3 - 9\,a^2 + 6\,a$ by $3\,x - 1 + 2\,a^2$.
10. Divide $a^4 - x^4$ by $a^3 - a^2\,x + a\,x^2 - x^3$.
11. Divide $6\,x^4 - 96$ by $3\,x - 6$.
12. Divide $4\,a^2 - a\,b$ by $2\,a - b$.
13. Divide $6\,a^4 + 9\,a^2 - 15\,a$ by $3\,a^2 - 3\,a$.

XXI. *Equations.*

The above rules are sufficient to solve all equations of the first degree.

1. Find the value of x in the equation

$$\frac{a\,b^2\,x - 2\,c}{5\,a} - \frac{2\,a\,c}{3\,a - b} = a\,b\,x - \frac{b^2\,x}{3}$$

First, clear it of fractions by multiplying by the denominators.

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* Let the learner prove his results by multiplication.

Expressing the multiplication, we have

$$(a b^2 x - 2 c)\,(3 a - b)\,(3) - (2 a c)\,(5 a)\,(3)$$
$$= (a b x)\,(5 a)\,(3 a - b)\,(3) - (b^2 x)\,(5 a)\,(3 a - b).$$

Performing the multiplication it becomes

$$9 a^2 b^2 x - 18 a c - 3 a b^3 x + 6 b c - 30 a^2 c$$
$$= 45 a^3 b x - 15 a^2 b^2 x - 15 a^2 b^2 x + 5 a b^3 x.$$

Transposing all the terms which contain x into the first member, and those which do not contain it into the second member, it becomes

$$9 a^2 b^2 x - 3 a b^3 x - 45 a^3 b x + 15 a^2 b^2 x + 15 a^2 b^2 x - 5 a b^3 x$$
$$= 18 a c - 6 b c + 30 a^2 c.$$

Uniting the terms which are alike

$$39 a^2 b^2 x - 8 a b^3 x - 45 a^3 b x = 18 a c - 6 b c + 30 a^2 c.$$

Separating the first member into factors

$$(39 a^2 b^2 - 8 a b^3 - 45 a^3 b)\, x = 18 a c - 6 b c + 30 a^2 c,$$

which gives $x = \dfrac{18 a c - 6 b c + 30 a^2 c}{39 a^2 b^2 - 8 a b^3 - 45 a^3 b}$.

2. Find the value of x in the following equation;

$$13 a - \frac{b x}{2 c} = 2 c x + d.$$

3. What is the value of x in the following equation?

$$\frac{2 a}{b - 3 x} + 4 b c = a b. \qquad \textit{Ans. } x = \frac{a b^2 - 2 a - 4 b^2 c}{3 a b - 12 b c}.$$

4. What is the value of x in the following equation?

$$\frac{7 x}{5 x - 2 a} - 13 b c = \frac{b^2 c - d}{2 b - 1}.$$

5. What is the value of x in the following equation?

$$\frac{7 a b x}{3 b c - 2 a x} + 2 b = \frac{a - 3 a}{1 - 5 b}.$$

XXII. *Miscellaneous Examples producing Simple Equations.*

1. A merchant sent a venture to sea and lost one fourth of it by shipwreck; he then added $2250 to what remained, and sent again. This time he lost one third of what he sent. He then added $1000 to what remained, and sent a third time, and gained a sum equal to twice the third venture; his whole return was equal to three times his first venture. What was the value of the first venture?

2. A man let out a certain sum of money at 6 per cent, simple interest, which interest in 10 years wanted but £12 to be equal to the principal. What was the principal?

3. A man let out £98 in two different parcels, one at 5, and the other at 6 per cent, simple interest; and the interest of the whole, in 15 years, amounted to £81. What were the two parcels?

4. A shepherd driving a flock of sheep in time of war, met a company of soldiers, who plundered him of one half the sheep he had and half a sheep over; the same treatment he received from a second, a third, and a fourth company, each succeeding company plundering him of one half the sheep he had left and one half a sheep over. At last he had only 7 sheep left. How many had he at first?

5. A man being asked how many teeth he had remaining, answered, three times as many as he had lost; and being asked how many he had lost, answered, as many as, being multiplied into $\frac{1}{6}$ part of the number he had left, would give the number he had at first. How many had he remaining, and how many had he lost?

After this question is put into equation every term may be divided by x.

6. There is a rectangular field whose length is to its breadth as 3 to 2, and the number of square rods in the field is equal to 6 times the number of rods round it. Required the length and breadth of the field.

7. What two numbers are those, whose difference, sum, and product, are to each other, as the numbers 2, 3, and 5 respectively?

8. Generalize the above by putting a, b, and c instead of 2, 3, and 5 respectively.

Let $x =$ the greater
and $y =$ the less.

Then

1. $x - y = \frac{a}{b}(x + y)$
2. $x - y = \frac{a}{c}xy$
3. by the first $y = \frac{bx - ax}{b + a} = \frac{b - a}{b + a}x$
4. by the 2nd $y = \frac{cx}{ax + c}$
5. by 3d and 4th $\frac{cx}{ax + c} = \frac{b - a}{b + a}x$
6. dividing by x $\frac{c}{ax + c} = \frac{b - a}{b + a}$
7. clearing of fractions $bc + ac = abx - a^2x + bc - ac$
8. by transposition $abx - a^2x = 2ac$
9. from the 8th $(b - a)x = 2c.$
10. $x = \frac{2c}{b - a}$
11. putting 10th into 3d $y = \frac{b - a}{b + a} \cdot \frac{2c}{b - a} = \frac{2c}{b + a}.$

Solve the 7th Ex. by these formulas; also try other numbers.

9. When a company at a tavern came to pay their reckoning, they found that if there had been three persons more, they would have had a shilling apiece less to pay; and if there had been two less, they would have had to pay a shilling apiece more. How many persons were there, and how much had each to pay?

10. A sum of money is to be divided equally among a certain number of persons. Now if there were three claimants less, each would receive 150 dollars more; and if there were 6 more, each would receive 120 dollars less. How many persons are there, and how much is each to receive?

11. What fraction is that, to the numerator of which if 1 be added, its value will be $\frac{1}{3}$, but if 1 be added to its denominator its value will be $\frac{1}{4}$.

12. What fraction is that, to the numerator of which if a be added, its value will be $\frac{m}{n}$, but if a be added to its denominator its value will be $\frac{p}{q}$?

Ans. Numerator $\frac{ap(m+n)}{mq-np}$,

Denominator $\frac{an(p+q)}{mq-np}$.

Solve the 11th example by these formulas.

13. What fraction is that, from the numerator of which if a be subtracted, its value will be $\frac{m}{n}$, but if a be subtracted from its denominator, its value will be $\frac{p}{q}$?

N. B. The answers to the 12th and 13th differ only in the signs of the denominators. The learner will find by endeavouring to solve particular examples from these formulas, that he will not always succeed. If in making examples for the 12th, he selects his numbers, so that np is greater than mq, the formula will fail; but if he takes the same numbers, and applies them according to the conditions of the 13th, they will answer those conditions. When mq is greater than np the numbers will not suit the conditions of the 13th, but they will answer to those of the 12th. The numbers in example 11th will not form an example according to the 13th. The following numbers will form an example for the 13th but not for the 12th.

14. What fraction is that, from the numerator of which if 3 be subtracted, its value will be $\frac{4}{5}$, but if 3 be subtracted from its denominator its value will be $\frac{7}{11}$?

The reason why numbers chosen indiscriminately will not satisfy the conditions of the above formulas will be explained hereafter.

Equations with several Unknown Quantities.

XXIII. *Questions involving more than two unknown Quantities.*

Sometimes it is necessary to employ, in the solution of a question, more than two unknown quantities. In this case, the question must furnish conditions enough to form as many distinct equations as there are unknown quantities.

1. A market woman sold to one man, 7 apples, 10 pears, and 12 peaches, for 63 cents; and to another, 13 apples, 6 pears, and 2 peaches, for 31 cents; and to a third, 11 apples, 14 pears, and 8 peaches for 63 cents. She sold them each time at the same rate. What was the price of each?

Let $x =$ the price of an apple,
$y =$ " a pear,
$z =$ " a peach.

Then we shall have

1. $7x + 10y + 12z = 63$
2. $13x + 6y + 2z = 31$
3. $11x + 14y + 8z = 63.$

The second being multiplied by 6, the z will have the same coefficient as in the first.

$$\begin{array}{ll} 4. & 78x + 36y + 12z = 186 \\ 1. & \underline{7x + 10y + 12z = 63} \\ 5. & 71x + 26y \quad * \quad = 123. \end{array}$$

If the second be multiplied by 4, the z will have the same coefficient as the 3d.

6. $52x + 24y + 8z = 124$
3. $11x + 14y + 8z = 63$

7. $41x + 10y \quad * \quad = 61$

We have now the two equations $71x + 26y = 123$
and $41x + 10y = 61$

which contain only two unknown quantities. These may now be reduced in the same manner as others with two unknown quantities.

Multiplying the 5th by 5, and the 7th by 13, the coefficient of y will be the same in both.

8. $355x + 130y = 615$
9. $533x + 130y = 793$

10. $178x \quad * \quad = 178$

We have now found an equation containing only one unknown quantity.

$$178x = 178$$
$$x = 1.$$

Putting the value of x into the 7th, it becomes

$$41 + 10y = 61$$
$$10y = 20$$
$$y = 2.$$

Putting the values of x and y into the 2d, it becomes

$$13 + 12 + 2z = 31$$
$$2z = 6$$
$$z = 3.$$

Ans. The apples 1, the pears 2. and the peaches 3 cents each.

In the same manner, questions, involving four unknown quantities, may be solved. First combine them two by two till one of the unknown quantities is eliminated from the whole, and there will be three equations with three unknown quantities. Then combine these three two by two, until one of the unknown quantities is eliminated, and then there will be two equations with two unknown quantities, and so on.

Either of the methods of elimination may be used as is most convenient.

It is not necessary that all the unknown quantities should enter into every equation.

2. A market woman sold at one time 7 eggs, 12 apples, and a pie for 26 cents; at another time 12 eggs, 18 pears, and 3 pies, for 69 cents; at a third time 20 pears, 10 apples, and 17 eggs for 69 cents; and at a fourth time, 7 pies, 18 apples, and 10 pears for 66 cents. Each article was sold, at every sale, at the same price as at first. What was the price of each article?

Let $u =$ the price of an egg,
$x =$ " an apple,
$y =$ " a pie,
$z =$ " a pear..

1. $7u + 12x + y = 26$
2. $12u + 18z + 3y = 69$
3. $17u + 20z + 10x = 69$
4. $10z + 18x + 7y = 66$
5. In the 1st, $y = 26 - 7u - 12x$.

Putting this value of y into the 2nd and 4th, they become

6. $12u + 18z + 78 - 21u - 36x = 69$
7. $10z + 18x + 182 - 49u - 84x = 66.$

Uniting and transposing terms

8. $18z - 9u - 36x = - 9$
9. $10z - 49u - 66x = - 116$
3. $20z + 17u + 10x = 69$

If the 9th be multiplied by 2, the coefficient of z will be the same as in the 3d;

10. $20z - 98u - 132x = - 232.$

Subtracting 10th from 3d

$$\begin{array}{ll} 3. & 20z + 17u + 10x = 69 \\ 10. & \underline{20z - 98u - 132x = -232} \\ 11.^{*} & {*} \quad 115u + 142x = 301 \end{array}$$

If the 8th be multiplied by 5, and the 9th by 9, the coefficients of z will be alike.

$$\begin{array}{ll} 12. & 90z - 45u - 180x = -45 \\ 13. & 90z - 441u - 594x = -1044. \end{array}$$

Subtracting 13th from 12th

$$\begin{array}{ll} 14. & 396u + 414x = 999. \end{array}$$

Deducing the value of x from 11th, and also from 14th.

$$\begin{array}{ll} 15. & x = \dfrac{301 - 115u}{142} \\ 16. & x = \dfrac{999 - 396u}{414}. \end{array}$$

Making these values of x equal, we have an equation containing only one unknown quantity.

$$\frac{999 - 396u}{414} = \frac{301 - 115u}{142}.$$

This equation solved in the usual way gives

$$u = 2$$

Putting this value of u into the 15th or 16th, we shall find

$$x = \frac{1}{2}.$$

Putting these values of x and u into the 1st, 2nd, or 4th, and we shall find

$$y = 6.$$

Putting the values of x and u into the 3d, and we shall find

$$z = 1\tfrac{1}{2}.$$

Ans. Eggs, 2 cents each, apples, ½ cent, pears, 1½ cent, and pies, 6 cents.

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* If the learner is at a loss how to subtract —233 from 69 let him transpose both into the first member, or some terms from the first to the second.

In this example, three different methods of elimination were employed. This was not necessary; either method might have been used for the whole. It is sometimes convenient to use one, and sometimes the other.

3. There are three persons, A, B, and C, whose ages are as follows; if B's age be subtracted from A's, the difference will be C's age; if five times B's age and twice C's age be added together, and from their sum A's age be subtracted, the remainder will be 147; the sum of all their ages is 96. What are their ages?

4. Three men, A, B, C, driving their sheep to market, says A to B and C, if each of you will give me 5 of your sheep, I shall have just half as many as both of you will have left. Says B to A and C, if each of you will give me 5 of yours, I shall have just as many as both of you will have left. Says C to A and B, if each of you will give me 5 of yours, I shall have just twice as many as both of you will have left. How many had each?

5. It is required to divide the number 72 into four such parts, that if the first part be increased by 5, the second part diminished by 5, the third part multiplied by 5, and the fourth part divided by 5, the sum, difference, product, and quotient, shall all be equal.

6. A grocer had four kinds of wine, marked A, B, C, and D. He mixed together 7 gallons of A, 5 gallons of B, and 8 gallons of C, and sold the mixture at $1.21 per gallon. He also mixed together 3 gallons of A, 10 of C, and 5 of D, and sold the mixture at $1.50 per gallon. At another time he mixed 8 gallons of A, 10 of B, 10 of C, and 7 of D, and sold the whole for $48. At another time he mixed together 18 gallons of A, and 15 of D, and sold the mixture for $48. What was the value of each kind of wine?

7. Find the values of u, x, y, and z, in the following equations.

$$x - 2y + 3z = 5u$$
$$3x - 15 - u = 4y - 23$$
$$2u + z - y = 27$$
$$y + 12 - 3x + 11u = 91.$$

8. Three persons, A, B, and C, talking of their money, says A to B and C, give me half of your money and I shall have a sum d; says B to A and C, give me one third of your money and I shall have d; says C to A and B, give me one fourth of your money, and I shall have d. How much had each?

XXIV. *Negative Quantities.*

It sometimes happens in the course of a calculation, through some misconception of the conditions of the question, that a quantity is added which ought to have been subtracted, or a quantity subtracted which ought to have been added. In this case, algebra will detect the error, and show how to correct it.

The length of a certain field is a, and its breadth b; how much must be added to its length, that its content may be c?

Let $x =$ the quantity to be added to the length.

Then $a + x =$ the length after adding x.

$$a\,b + b\,x = c$$

$$b\,x = c - a\,b$$

$$x = \frac{c}{b} - a.$$

Suppose the length to be 8 rods, and the breadth 5; how much must be added to the length, that the field may contain 60 square rods?

Here $a = 8, b = 5$, and $c = 60$

$$x = \frac{60}{5} - 8 = 4.$$

Ans. 4 rods, and the whole length will be 12 rods.

Suppose the length 8 rods, and the breadth 5; how much must be added to the length, that the field may contain 30 square rods?

$$x = \frac{30}{5} - 8 = -2.$$

The answer is -2 rods. What shall we understand by this negative sign?

Let us return to the original equation.

$$8 \times 5 + 5x = 30$$
$$\text{or} \qquad 40 + 5x = 30.$$

Here appears an absurdity in supposing something to be added to 40 to make 30. The result shows that we must add -2 rods, that is, subtract 2 rods, which is in fact the case; for

$$40 - 5 \times 2 = 30.$$

Let the question be proposed as follows. There is a field 8 rods long and 5 wide; how much must be subtracted from the length, that the field may contain 30 square rods?

$$40 - 5x = 30$$
$$x = 2.$$

The value of x is now positive, which shows that the question is correctly expressed.

There is a field 8 rods long and 5 rods wide, how much must be subtracted from the length, that the field may contain 50 square rods?

$$40 - 5x = 50$$
$$x = -2.$$

Here again the value of x is negative, which shows some inconsistency in the question.

The inconsistency consists in supposing that something must be subtracted from 40 to make 50. In order to correct it, suppose something added. That is, put into the equation $+ 5x$ instead of $- 5x$.

Hitherto we have treated of negative quantities only in connexion with positive. They arise from the necessity of expressing subtraction by a sign, because it cannot actually be performed on dissimilar quantities. They are only positive quantities subtracted, and in their nature they differ in nothing from positive quantities. In that connexion we discovered rules for operating upon the quantities affected with the sign $-$.

It may sometimes happen as we have just seen, that by some wrong supposition in the conditions of the question, the quantities to be subtracted may become greater than those from which they are to be subtracted, in which case the whole expression taken together, or which is the same thing, the result after subtraction, will be negative. This is what is properly called a *negative quantity*.

A negative quantity cannot in reality be a quantity less than nothing, but it implies some contradiction. It answers to a figure of speech frequently used. If it is asked, how much a man is worth who owes five thousand dollars more than he can pay, we sometimes say he is worth five thousand dollars less than nothing, instead of changing the form of expression and saying, he owes five thousand dollars more than he can pay.

If any thing is added to a number, properly speaking it must increase the number; if we add nothing, it is not altered. It is impossible to add less than nothing; but by a figure of speech we may use the expression, *add a quantity less than nothing*, to signify subtraction.

As these negative quantities may frequently occur, it is necessary to find rules for using them.

In the first place, let us observe, that all negative quantities are derived from endeavouring to subtract a larger quantity from a smaller one. The largest number that can actually be subtracted from any number, is the number itself. Thus the largest number that can be subtracted from 5 is 5; the largest number that can be subtracted from a is a itself. If it be required to subtract 8 from 5, it becomes $5 - 5 - 3 = -3$; the 5 only can be subtracted, the 3 remains with the sign $-$, which shows that it could not be subtracted. If 5 be subtracted from 8, the remainder is 3, the same as in the other case except the sign.

In the same manner, if it be required to subtract b from a, b being the larger the remainder will have the sign $-$, that is, $a - b$ will be a negative quantity.

Suppose $b - a = m$; then $a - b = -m$. That is, whether a be subtracted from b or b from a, the numerical value of the remainder is the same, differing only with respect to the sign.

It is required to add the quantity $a - b$ to c.

The answer is evidently $c + a - b$.

Now if a is greater than b, the quantity $c + a - b$, is greater than c, by the difference between a and b; but if b is greater than a, the quantity is smaller than c, by the difference between a and b. That is, if

$$b - a = m,$$

then $\quad a - b = -m$

and $\quad c + a - b = c - m.$

Hence, adding a negative quantity, is equivalent to subtracting an equal positive quantity.

In the above example of the field, in which the length was 8 rods and breadth 5, it was asked, how much must be added to the length, that it might contain 30 square rods. The answer was -2; which was equivalent to saying, you must subtract 2 rods.

It is required to subtract $a - b$ from c.

The answer is evidently $c - a + b$.

Now if a is greater than b, the quantity $c - a + b$ is less than c by the difference between a and b, but if b is greater than a, the quantity is larger than c, by the same quantity.

Let $\quad a - b = -m \quad$ which gives $\quad -a + b = m$

then $\quad c - a + b = c + m.$

Hence, subtracting a negative quantity, is equivalent to adding an equal positive quantity.

In the example of the field, in which the length was 8 rods and the breadth 5, it was asked, how much must be subtracted from the length, that the field might contain 50 square rods.

The answer was -2 rods, which was equivalent to saying, that 2 rods must be added to the length.

A is worth a number a of dollars, B is not worth so much as A by a number b of dollars, and C is worth c times as much as B. How much is C worth?

B's property $= a - b.$

C's property $= a\,c - b\,c.$

Now if a is greater than b, the quantity $a\,c - b\,c$ will be positive; but if b is greater than a, then $a - b$ is negative, and also $a\,c - b\,c$ is negative.

Let $\quad b - a = m.$

then $\quad b\,c - a\,c = c\,m.$

and $\quad a\,c - b\,c = -c\,m.$

or $\quad c\,(a - b) = -c\,m.$

That is, if B is in debt, C is c times as much in debt. Hence if a negative quantity be multiplied by a positive, the product is negative.

A gentleman owned a number a of farms, and each farm was worth a number c of dollars, which was his whole property. He hired money and fitted out a number b of vessels, and each vessel was worth as much as one of his farms. All the vessels were lost at sea. How much was he then worth.

He was worth $a - b$ times c dollars. That is, $a\,c - b\,c$ dollars.

Now if the number of farms exceeded the number of vessels, he still had some property, but if the number of vessels exceeded the number of farms, (that is, if b is larger than a,) the quantity $a\,c - b\,c$ is negative, and he owed more than he could pay.

Hence if a positive quantity be multiplied by a negative the product will be negative.

Multiply $a - b$ by $c - d$.

$$\begin{array}{l} a - b \\ c - d \\ \hline \end{array}$$

Product $a\,c - b\,c - a\,d + b\,d$.

This product may be put in this form.

$$(a - b)\,c + (b - a)\,d.$$

Let it be remembered that $a - b$ has the same numerical value as $b - a$, they differ only in the sign.

Suppose $a - b = -m$

by changing all the signs $b - a = +m$.

Hence $(a - b)\,c + (b - a)\,d = -c\,m + d\,m = m\,(d - c)$.

Now if d is greater than c, (which is the case when $c - d$ is negative,) the quantity $m\,(d - c)$ is positive.

Hence if a negative quantity be multiplied by a negative, the product will be positive.

Another demonstration. Suppose both $a - b$ and $c - d$ to be negative, as before; then $b - a$ and $d - c$ will both be positive, and their product will be positive.

$$\begin{array}{l} b - a \\ d - c \\ \hline bd - bc - ad + ac. \end{array}$$

This product is precisely the same as that produced by multiplying $a - b$ by $c - d$. Therefore if two negative quantities be multiplied together, the product will be the same as that of two positive quantities of the same numerical value, and will have the positive sign.

It is required to find the second power of $a - b$, and also of $b - a$.

The second power of each is $a^2 + b^2 - 2ab$.

Now if $a - b$ is positive, then $b - a$ is negative; or if $a - b$ is negative, then $b - a$ is positive.

Suppose $a - b = m$

then $b - a = -m$

we have $(a - b)^2 = (b - a)^2 = m^2$.

That is, the second power of any quantity, whether positive or negative, is necessarily positive.

The rules for division will necessarily follow from those of multiplication.

Hence the rules which apply to terms affected with the sign $-$ in compound quantities, extend to isolated negative quantities.

We might also derive the same rules in the following manner. It has been shown that a negative quantity is derived from some contradiction in the conditions of question, by which that quantity entered into the equation with the wrong sign. Now, in order to make it right, the sign of that quantity must be changed in all places where it is used. That is, if it was before added, it must now be subtracted; and if it was subtracted before, it must now be added, and that whether multiplied by another quantity or not.

Suppose we have the equation

$$ax - 2x^2 - 2abx = c - d.$$

Now suppose that we have $x = -m$.

This shows that x was used in all cases with the wrong sign, therefore to insert $-m$ in place of x we must change the sign in each term where x is found.

Take the quantity first without x, thus,

$$a - 2 - 2ab.$$

First insert $-m$ in the second term and it becomes

$$a + 2m - 2ab.$$

Now insert $-m$ into all the terms, and it becomes

$$-am - 2m^2 + 2abm = c - d.$$

If $-m$ be inserted by the rules found above, the same result will be produced.

When a negative value has been found for the unknown quantity, we have observed it shows that there was some inconsistency in the question. If then the unknown quantity be put again into the same equation, with the contrary sign, as we introduced $-m$ above, that is, if the unknown quantity be taken with the negative sign, and introduced by the above rules into all the terms where it was found before, a new equation will be produced, differing from the former only in some of the signs. Then if the conditions of the question be altered so as to correspond with the new equation, it will be consistent, and a positive value will be obtained for the unknown quantity. The new value of the unknown quantity however will be the same as the former, with the exception of the sign. Therefore, when once we are accustomed to interpret this kind of results, it will be unnecessary to go through the calculation a second time.

The following examples are intended to exercise the learner in interpreting these results.

1. A father is 55 years old, and his son is 16. In how many years will the son be one fourth as old as the father?

Let $x =$ the number of years.

$$16 + x = \frac{55 + x}{4}$$

$$64 + 4x = 55 + x$$

$$3x = 55 - 64 = -9$$

$$x = -3.$$

Here x has a negative value, consequently it entered into the equation with the wrong sign. Putting now $-x$ instead of x into the equation, it becomes

$$16 - x = \frac{55 - x}{4}.$$

This shows that something must be subtracted from the present age; that is, the son was a fourth part as old as the father some years before.

This equation gives

$$x = 3.$$

Therefore he was one fourth part as old 3 years before, when the father was 52, and the son 13.

2. A man when he was married was 45 years old, and his wife 20. How many years before, was he twice as old as she?

$$20 - x = \frac{45 - x}{2}$$
$$x = -5.$$

There is a wrong supposition in this question. Putting $-x$ into the equation it becomes

$$20 + x = \frac{45 + x}{2}$$
$$x = 5.$$

This shows that she was not half as old as he when they were married, but that it was to happen 5 years afterward, when the man was 50, and the wife 25.

3. A labourer wrought for a man 15 days, and had his wife and son with him the first 9 days, and received $14.25. He afterwards wrought 12 days, having his wife and son with him 5 days, and received $13.50. How much did he receive per day himself, and how much for his wife and son?

4. A labourer wrought for a man 11 days, and had his wife with him 4 days, and received $17.82. He afterwards wrought 23 days, having his wife with him 13 days, and received $38.78. How much did he receive per day for himself, and how much for his wife?

5. A labourer wrought for a gentleman 7 days, having his wife with him 4 days, and his son 3 days, and received \$7.80. At another time he wrought 10 days, having his wife with him 7 days, and his son 5 days, and received \$11.65. At a third time he wrought 8 days, having his wife with him 5 days, and his son 8 days, and received \$7.54. How much did he receive per day himself, and how much for his wife and son severally?

6. What number is that, whose fourth part exceeds its third part by 16?

$$\frac{x}{4} = \frac{x}{3} + 16$$
$$x = -192.$$

The question as it was proposed involves some contradiction. Putting in $-x$ it becomes

$$-\frac{x}{4} = -\frac{x}{3} + 16.$$

Changing all the signs

$$\frac{x}{4} = \frac{x}{3} - 16$$
$$x = 192.$$

This shows that the question should have been as follows; What number is that, whose third part exceeds its fourth part by 16?

7. What number is that, $\frac{7}{11}$ of which exceeds $\frac{4}{7}$ of it by 18?

8. What fraction is that, to the numerator of which if 1 be added, its value will be $\frac{3}{5}$, but if 1 be added to its denominator, its value will be $\frac{4}{7}$?

9. What fraction is that, from the denominator of which, if 2 be subtracted, its value will be $\frac{14}{15}$, but if 2 be subtracted from its numerator, its value will be $\frac{1}{2}$?

10. It is required to divide the number 20 into two such parts, that if the larger be multiplied by 3, and the smaller by 5, the sum of the products will be 125.

11. It is required to find two numbers, whose difference is 25, and such that if the larger be multiplied by 7, and the smaller by 5, the difference of their products shall be 215?

XXV. *Explanation of Negative Exponents.*

It was observed above, that when the dividend and the divisor were different powers of the same letter, division is performed by subtracting the exponent of the divisor from that of the dividend: thus

$$\frac{a^7}{a^3} = a^{7-3} = a^4.$$

Now $\frac{a}{a} = 1$. By the above principle $\frac{a}{a} = a^{1-1} = a^0$; therefore $a^0 = 1$.

Also $\frac{a^3}{a^3} = a^{3-3} = a^0 = 1$; $\frac{b}{b} = b^{1-1} = b^0 = 1$;

$\frac{10}{10} = 10^{1-1} = 10^0 = 1$; $\frac{a+b}{a+b} = (a+b)^{1-1}$

$= (a+b)^0 = 1$[*].

That is, any quantity having zero for its exponent, is equal to 1.

Again $\frac{a}{a^2} = \frac{1}{a}$, or $\frac{a}{a^2} = a^{1-2} = a^{-1}$

$$\frac{a^3}{a^7} = a^{3-7} = a^{-4} = \frac{1}{a^4}$$

Hence it appears that a^{-1} has the same value as $\frac{1}{a}$, and a^{-4} as $\frac{1}{a^4}$.

The quantities a^3, a^2, a^1, a^0, a^{-1}, a^{-2}, a^{-3}, &c. have the same value as a^3, a^2, a^1, 1, $\frac{1}{a}$, $\frac{1}{a^2}$, $\frac{1}{a^3}$, &c.

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[*] Exponents may be used for compound quantities as well as for simple; and multiplication and division may be performed on those which are similar, by adding and subtracting the exponents.

On this principle the denominator of a fraction, or any factor of the denominator may be written in the numerator by giving its exponent the sign —. This mode of notation is often very convenient; I shall therefore give a few examples of its application.

$$\frac{2a}{bc^2} = \frac{2ab^{-1}}{c^2} = 2ab^{-1}c^{-2}.$$

1. Multiply $\frac{2a}{bc^2}$ by b^3c.

By the common rule $\frac{2a}{bc^2} \times b^3c = \frac{2ab^3c}{bc^2} = \frac{2ab^2}{c}.$

By the principle explained above,

$$2ab^{-1}c^{-2} \times b^3c = 2ab^{-1+3}c^{-2+1} = 2ab^2c^{-1} = \frac{2ab^2}{c}.$$

2. Multiply $3ac^{-3}d^{-2}$ by $3a^2c^4d^4$.

3. Multiply $5a^{-2}c^{-3}$ by $2ac^7$.

4. Multiply $\frac{13bd}{2a^5c^2}$ by $3a^3c^5$.

5. Multiply $2a(b+d)^{-3}$ by $3a(b+d)^3$.

6. Multiply $\frac{3ab}{4c(2a-bd)^2}$, by $8c^3(2a-bd)^3$

7. Divide $\frac{3a}{c^3}$ by c^2.

By the common method $\frac{3a}{c^3} \div c^2 = \frac{3a}{c^5}.$

By the above method $3ac^{-3} \div c^2 = 3ac^{-3-2}$

$$= 3ac^{-5} = \frac{3a}{c^5}$$

Or thus, to divide $3ac^{-3}$ by c^2, is the same as to multiply it by $\frac{1}{c^2}$ or c^{-2}, which gives the same result.

8. Divide $\frac{5ac}{(2b-c)^2}$ by $a^2(2b-c)^2$.

9. Multiply $\frac{3a}{cd}$ by $\frac{4e}{c^3d}$.

$$\frac{3a}{cd} = 3ac^{-1}d^{-1} \text{ and } \frac{4e}{c^3d} = 4ec^{-3}d^{-1}.$$

$$3ac^{-1}d^{-1} \times 4ec^{-3}d^{-1} = 12ac^{-4}d^{-2}e = \frac{12ae}{c^4d^2}.$$

10. Divide $\frac{2a}{c^2d}$ by $\frac{3b}{a^2c^2d}$.

$$2ac^{-2}d^{-1} \div 3ba^{-2}c^{-2}d^{-1}$$

$$= \frac{2a^{1+2}b^{-1}c^{-2+2}d^{-1+1}}{3} = \frac{2a^3b^{-1}}{3} = \frac{2a^3}{3b}.$$

In this example the exponents to be subtracted had the sign —, which in subtracting was changed to +.

11. Multiply $\frac{3a(bc-d)^3}{2bc}$ by $\frac{7am}{3(bc-d)^5}$.

12. Multiply $\frac{4ab^3}{3c(b-2c)^3}$ by $\frac{bc^2(b-2c)^7}{12a^2b^3}$.

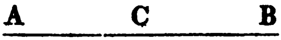
13. Divide $\frac{15a^3(bc-2)^3}{16b^2c^3}$ by $\frac{45a^3(bc-2)^2}{24b^2c^3}$.

14. Divide $\frac{a^3(17b+3d)^7}{(a-b)^5}$ by $\frac{5a^2(17b+3d)^5}{4(a-b)^5}$.

XXVI. *Examination of General Formulas.*

When a question has been resolved generally, that is, by representing the known quantities by letters, we sometimes propose to determine what values the unknown quantities will take, for particular suppositions made upon the known quantities.

The two following questions offer nearly all the circumstances that can ever occur in equations of the first degree.



Two couriers set out at the same time from the points A and B, distant from each other a number m of miles, and travel towards each other until they meet. The courier who sets out from the point A, travels at the rate of a miles per hour; the other travels at the rate of b miles per hour. At what distance from the points A and B will they meet?

Suppose C to be the point, and
Let $x =$ the distance A C
and $y =$ the distance B C.
For the first equation we have

$$x + y = \text{A B} = m$$

Since the first courier travels x miles, at the rate of a miles per hour, he will be $\frac{x}{a}$ hours upon the road. The second courier will be $\frac{y}{b}$ hours upon the road. But they travel equal times; therefore,

$$\frac{x}{a} = \frac{y}{b}$$

$$x = \frac{ay}{b}.$$

Putting this value of x into the first equation, it becomes

$$\frac{ay}{b} + y = m$$

$$ay + by = bm$$

$$y = \frac{bm}{a+b}$$

$$x = \frac{ay}{b} = \frac{a}{b} \times \frac{bm}{a+b} = \frac{abm}{b(a+b)} = \frac{am}{a+b}.$$

Since neither of the quantities in these values of x and y has the sign $-$, it is impossible for either value to become negative. Therefore whatever numbers may be put in place of a, b, and m, they will give an answer according to the conditions of the question. In fact, since they travel towards each other, whatever be the distance of the places, and at whatever rate they travel, they must necessarily meet.

Suppose now that the two couriers setting out from the points A and B situated as before, both travel in the same direction towards D, at the same rates as before. At what distances from the points A and B will the place of their meeting, C, be?

Let $x =$ the distance from A to C,
and $y =$ " B to C.

$$x - y = AC - BC = AB = m.$$

The second equation expressing only the equality of the time will not be altered.

$$\frac{x}{a} = \frac{y}{b}$$

Solving the two equations as before,

$$x = \frac{ay}{b}.$$

$$\frac{ay}{b} - y = m$$

$$ay - by = bm$$

$$y = \frac{bm}{a-b}$$

$$x = \frac{a}{b} \times y = \frac{a}{b} \cdot \frac{bm}{a-b} = \frac{abm}{b(a-b)} = \frac{am}{a-b}.$$

Here the values of x and y will not be positive unless a is greater than b; that is, unless the courier, that sets out from A, travels faster than the other.

Suppose $a = 8$ and $b = 4$.

Then $$x = \frac{8m}{8-4} = \frac{8m}{4} = 2m$$

$$y = \frac{4m}{8-4} = \frac{4m}{4} = m.$$

In this case the point C, where they come together, is distant from A twice the distance A B.

Suppose a smaller than b, for example

$$a = 4 \text{ and } b = 8.$$

Then $$x = \frac{4m}{4-8} = -m$$

$$y = \frac{8m}{4-8} = -2m.$$

Here the values of x and y are both negative; hence there is some absurdity in the enunciation of the question for these numbers. In fact, it is impossible that the courier setting out from A, and travelling slower than the other should overtake him.

Let us put x and y negative in the two equations, that is, change their signs.

They become $$-x + y = m$$

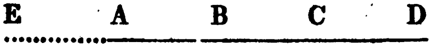
$$-\frac{x}{a} = -\frac{y}{b}$$

or $$y - x = m$$

and $$\frac{x}{a} = \frac{y}{b}.$$

The second equation is not affected by changing the sign; and it ought not to be so, since it expresses only the equality of the times.

The first equation becomes $y - x = m$, instead of $x - y = m$, which shows that the point where they are together is nearer to A than to B, by the distance from A to B. It must therefore be on the other side of A, as at E.



The enunciation of the question may be changed in two ways so as to answer the conditions of this equation.

First, we may suppose, that the couriers, setting out from A and B, instead of going towards D, go in the opposite direction, the one from A at 4 miles per hour, and the other from B at 8 miles per hour; at what distance from the points A and B is the point E, where they come together?

Or we may suppose that two couriers setting out from the same place E, one travelling at the rate of 4 miles, and the other 8 per hour, have arrived at the same time at the points A and B, which are m miles asunder. What distance are the points A and B from E?

Suppose $a = b$.

Then $$x = \frac{am}{a-b} = \frac{am}{a-a} = \frac{am}{0}$$

$$y = \frac{bm}{a-b} = \frac{bm}{a-a} = \frac{am}{0}.$$

How is this result to be interpreted?

Observe that in this case a and b being equal, the two couriers travel equally fast, it is therefore impossible that one should ever overtake the other, however far they may travel in either direction, and no change in the conditions can make it possible. Zero being divisor, then, is a sign of *impossibility*.

We may observe that when there is any difference, however small, between a and b, the values of x and y will be real, and the couriers will come together in one direction or the other; and the smaller the difference, the greater will be the distance travelled before they come together; that is, the greater will be the values of x and y.

Suppose $a = 5$ and $b = 4$; $a - b = 1$,

then $$x = \frac{5m}{1} = 5m \quad y = \frac{4m}{1} = 4m.$$

Again, Suppose $a = 5$, and $b = 4 \cdot 5$, $a - b = \cdot 5$,

then $$x = \frac{5m}{\cdot 5} = 10m \quad y = \frac{4 \cdot 5m}{\cdot 5} = 9m.$$

Again, Suppose $a = 5$, and $b = 4 \cdot 98$, $a - b = \cdot 02$,

then $$x = \frac{5\,m}{\cdot 02} = 250\,m, \text{ and } y = \frac{4 \cdot 98}{\cdot 02} = 249\,m$$

Again, Suppose $a = 5$ and $b = 4{\cdot}998$, $a - b = {\cdot}002$,

then $$x = \frac{5\,m}{\cdot 002} = 2500\,m.$$

and $$y = \frac{4{\cdot}998\,m}{2} = 2499\,m.$$

Here observe, that as the difference between a and b becomes very small, the values of x and y become very large, and the difference between them is always m. Hence, since the smaller the divisor the larger the quotient, we may conclude, that when the divisor is actually zero, the quotient must be infinite. From this consideration, mathematicians have called the expression $\frac{a}{0}$, that is, a quantity divided by zero, *a symbol of infinity*. They therefore say, that, both couriers travelling equally fast, the distance, travelled before they come together, is infinite. But as infinity is an impossible quantity, I prefer the term *impossible*, as being a term more easily comprehended. I shall therefore call $\frac{a}{0}$ a symbol of *impossibility.*

If a quantity be divided by an infinite or impossible quantity, the quotient will be zero. If b be divided by $\frac{a}{0}$, it becomes $\frac{b}{\frac{a}{0}}$. Multiply both numerator and denominator by 0, it becomes $\frac{0 \times b}{a} = 0$. In fact, since the larger the divisor, the smaller the quotient, the dividend remaining the same, it follows that if the divisor surpasses any assignable quantity, the quotient must be smaller than any assignable quantity, or nothing.

One case more deserves our notice. It is when $a = b$ and $m = 0$; in which case we have

$$x = \frac{am}{a-b} = \frac{a \times 0}{0} = \frac{0}{0}$$

$$y = \frac{bm}{a-b} = \frac{b \times 0}{0} = \frac{0}{0}.$$

If we return to the equations themselves, they become

$$x - y = 0$$

$$\frac{x}{a} = \frac{y}{a}.$$

From the first we have

$$x = y$$

Substituting this value in the second

$$\frac{y}{a} = \frac{y}{a}.$$

This last equation has both its members alike, and is sometimes called an *identical equation.* The values of the unknown quantities cannot be determined from it. In fact, since m is zero, both couriers set out from the same point. And since they both travel at the same rate, they are always together. Therefore there is no point where they can be said to come together. The expression $\frac{0}{0}$ is here an expression of an *indeterminate* quantity.

There are some cases where an expression of this kind is not a sign of an indeterminate quantity, but in these cases it arises from a factor being common to the numerator and denominator, which by some suppositions becomes zero, and renders the fraction of the form of $\frac{0}{0}$; but being freed from that factor, it has a determinate value.

The following expression is an example of it.

$$\frac{a\,(a^2 - b^2)}{b\,(a - b)}.$$

When $a = b$, this expression becomes $\frac{0}{0}$. But both numerarator and denominator contain the factor $a - b$, which becomes zero when a and b are equal.

Dividing by $a - b$, the expression becomes

$$\frac{a\,(a+b)}{b},$$

which is equal to $2\,a$ when $a = b$.

It is necessary then, when we find an expression of the form $\frac{0}{0}$, before pronouncing it an indeterminate quantity, to see if there is not a factor, common to the numerator and denominator, which, becoming zero, renders the expression of this form.

The example of the couriers furnishes some other curious cases, for which we must refer the learner to Lacroix's or Bourdon's Algebra.

Let the learner examine the following examples in a similar manner.

In Art. IX. examples 15 and 16, the following formulas, relating to interest, were obtained. How are r and t to be interpreted, when p is greater than a; and how when a and p are equal?

$$r = \frac{a-p}{t\,p}, \qquad t = \frac{a-p}{r\,p}.$$

In Art. XXII. examples 12th and 13th, the following formulas were obtained. In what cases will the results become negative, and how are the negative results to be interpreted?

12th. Numerator $\frac{a\,p\,(m+n)}{m\,q - n\,p}$.

Denominator $\frac{a\,n\,(p+q)}{m\,q - n\,p)}$.

13th. Numerator $\frac{a\,p\,(m+n)}{n\,p - m\,q}$

Denominator $\frac{a\,n\,(p+q)}{n\,p - m\,q)}$.

It is required to divide a given number a into two such parts, that if r times one part be added to s times the other part, the sum will be a given number b.

Ans. The part to be multiplied by r is $\frac{b - a\,s}{r - s}$,

and the part to be multiplied by s is $\frac{a\,r - b}{r - s}$.

In what cases will one or both of these results be negative? Can both be negative at the same time? How are the negative results to be interpreted? In what cases will either of them become zero? Can both become zero at the same time? What is to be understood when one or both become zero? In what cases, will one or both become infinite or impossible? Can either of them ever be of the form $\frac{0}{0}$?

XXVII. *Equations of the Second Degree.*

1. A boy being asked how many chickens he had, answered, that if the number were multiplied by four times itself, the product would be 256. How many had he?

Let $x =$ the number,
then $4\,x =$ four times the number.

$$4\,x \times x = 4\,x^2$$

By the conditions $4\,x^2 = 256$

$$x^2 = 64$$

That is $x\,x = 64.$

This equation is essentially different from any which we have hitherto seen.

It is called an equation of the *second degree*, because it contains x^2, or the second power of the unknown quantity. In order to find the value of x, it is necessary to find what number, multiplied by itself, will produce 64. We know immediately by the table of Pythagoras that $8 \times 8 = 64$. Therefore

$x = 8.$ *Ans.* 8 chickens.

Note. The results of these equations may be proved like those of the first degree.

2. A boy being asked his age, answered, that if it were multiplied by itself, and from the product 37 were subtracted, and the remainder multiplied by his age, the product would be 12 times his age. What was his age?

$$x \times x = x^2 \quad (x^2 - 37)\,x = x^3 - 37\,x.$$

By the conditions

$$x^3 - 37\,x = 12\,x.$$

Dividing by x,

$$x^2 - 37 = 12$$

$$x^2 = 49$$

$$x = 7.$$

Ans. 7 years.

3. There are two numbers in the proportion of 5 to 4, and the difference of whose second powers is 9. What are the numbers?

Let $x =$ the larger number,

then $\frac{4\,x}{5} =$ the smaller.

The second power of $\frac{4x}{5}$ is $\frac{16\,x^2}{25}$.

By the conditions $x^2 - \frac{16\,x^2}{25} = 9$.

4. There are two numbers whose sum is to the less in the proportion of 15 to 4, and whose sum multiplied by the less produces 135. What are the numbers?

Let $x =$ the less, and $y =$ the greater.

Then $x + y = \frac{15\,x}{4}$

and $x\,(x + y) = 135$.

The second gives $y = \frac{135 - x^2}{x}$.

Putting this value of y into the first, it becomes

$$x + \frac{135 - x^2}{x} = \frac{15\,x}{4}, \text{\&c.}$$

Hence it appears, that when an example involves the second power of the unknown quantity, the value of the second power must first be found in the same manner as the unknown quantity is found in simple equations; and from the value of the second power, the value of the first power is derived.

It is easy to find the second power of any quantity, when the first power is known, because it is done by multiplication; but it is not so easy to find the first power from the second. It cannot be done by division, because there is no divisor given. When the number is the second power of a small number, the first power is easily found by trial, as in the above examples. When the number is large, it is still found by trial; but a rule may be very easily found, by which the number of trials will be reduced to very few. The first power is called the *root* of the second power, and when it is required to find the first power from the second, the process is called *extracting the root.*

It has been shown, Art. XXIV. that the second power of every quantity, whether positive or negative, is necessarily positive; thus $3 \times 3 = +9$, and also $-3 \times -3 = +9$. So $a \times a = a^2$, and also $-a \times -a = a^2$. Hence every second power, properly speaking, has two roots, the one positive and the other negative. The conditions of the question will generally show which is the true answer.

XXVIII. *Extraction of the Second Root.*

In order to find a rule for extracting the root, or finding the first power from the second, it will be necessary, first, to observe how the second power is formed from the first.

Let $a = 20$ and $b = 7$; then $a + b = 27$.

The second power of $a + b$ is

$$(a + b)\,(a + b) = a^2 + 2\,a\,b + b^2.$$

$$\begin{aligned} a^2 &= 20 \times 20 = 400 \\ ab &= 20 \times \ \ 7 = 140 \\ ab &= 20 \times \ \ 7 = 140 \\ b^2 &= \ \ 7 \times \ \ 7 = \ \ 49 \\ a^2 + 2\,a\,b + b^2 &= 729. \end{aligned}$$

The product is formed in precisely the same manner in the usual mode of multiplication, as may be seen, if the products are written down as they are formed, without carrying.

```
  27
  27
 ———
  49
 140
 140
 400
————
 729
```

Here we observe, 7 times 7 is 49, 7 times 20 is 140, 20 times 7 is 140, and lastly 20 times 20 is 400. These added together make 729, which is the second power of 27.

We observe,

1st. When the root or first power consists of two figures, the second power consists of the second power of the tens, plus the product of twice the tens by the units, plus the second power of the units.

2d. The second power of 9, the largest number consisting of one figure, is 81 ; and the second power of 10, the smallest number consisting of two places, is 100 ; and the second power of 100, the smallest number consisting of three places, is 10000. Hence, when the root consists of one figure, the second power cannot exceed two figures ; and when the root consists of two figures, the second power consists of not less than three figures, nor more than four figures.

From these remarks it appears, that we must first endeavour to find the second power of the tens, and that it will be found among the hundreds and thousands.

Let it be required to find the root of 729. This number contains hundreds, therefore the root will contain tens. The second power of the tens is contained in the 700. 20×20 is 400, and 30×30 is 900. 400 is the greatest second power of tens contained in 700. The root of 400 is 20. Subtract 400 from 729, and the remainder is 329. This must contain $2\,a\,b + b^2$, that is, the product of twice the tens by the units, plus the second power of the units. If it contained exactly the product $2\,a\,b$ of twice the tens by the units, the units of the root would be found by dividing 329 by twice 20, or 40; for $2\,a\,b$ divided by $2\,a$ gives b. As it is, if we divide by twice 20 or 40, we shall obtain a quotient either exact, or too large by 1 or 2. 40 is contained in 329, 8 times. Write 8 in the root and raise the whole to the second power. 28 × 28 = 784, which is larger than 729. Next try 7 in the place of 8. 27 × 27 = 729. Therefore 7 is right, and 27 is the root required.

The operation may stand as follows.

$$\begin{array}{l} 729\ (20 + 7 = 27 \text{ root.} \\ 400 \\ \hline 329\ (40 \text{ divisor.} \end{array}$$

27 × 27 = 729.

What is the root of 1849?

$$\begin{array}{l} 18{,}49\ (40 + 3 = 43 \text{ root.} \\ 16{,}00 \\ \hline \ \ 249\ (80 \text{ divisor.} \end{array}$$

43 × 43 = 1849.

In this example, the second power of the tens will be found in the 1800. 30 × 30 = 900; 40 × 40 = 1600; 50 × 50 = 2500. The greatest second power in 1800 is 1600, the root of which is 40. Write 40 in the place of a quotient. Subtract 1600 from 1829. The remainder is 249, which divided by twice 40, or 80, gives 3. Add 3 to the root, and raise the whole to the second power. 43 × 43 = 1849. Therefore 43 is the root required.

It is evident that the result will not be affected, if instead of writing 40 in the root at first, we omit the zero, and then subtract the second power of 4, viz. 16 from the 18, omitting the two zeros which come under the other period. Then to form the divisor, the 4 may be doubled, and the divisor will be 8 instead of 80, and the dividend must be 24, the right hand figure being rejected.

Operation.

```
                  18,49 (43 root.
                  16
                  ─────
Dividend =         24,9 (8 divisor.
 43 × 43 =         18 49.
```

Examples.

1. What is the root of 1444? *Ans.* 38.
2. What is the root of 7396?
3. What is the root of 361?
4. What is the root of 3249?
5. What is the root of 7921?
6. What is the root of 8281?

57
×9

The second power of $a + b + c$, or $(a + b + c)(a + b + c)$ is

$a^2 + 2ab + b^2 + 2ac + 2bc + c^2 = \ldots\ldots\ldots\ldots$

$a^2 + 2ab + b^2 + 2(a + b)c + c^2$.

To find the second power of 726

Let $a = 700$, $b = 20$, and $c = 6$.

$$\begin{array}{rlr}
a^2 = & 700 \times 700 & = 490000 \\
2ab = & 2 \times 700 \times 20 & = 28000 \\
b^2 = & 20 \times 20 & = 400 \\
2(a + b)c = & 2 \times (700 + 20) \times 6 & = 8640 \\
c^2 = & 6 \times 6 & = 36 \\
\hline
 & & 527076
\end{array}$$

```
   726
   726
  ────
  4356
 1452
5082
──────
527076
```

The first three terms of the formula, viz.

$$a^2 + 2\,a\,b + b^2,$$

are the second power of $a + b$ or of the hundreds and tens, viz. 720. The second power of 720 can have no significant figure below hundreds, and the significant figures of the second power of 720 and of 72 are the same; the former is 518400, the latter 5184. If from the whole number 527076 the two right hand figures be rejected, the number is 5270. This contains the second power of 72 and something more, viz. a part of the product $2 \times (700 + 20) \times 6 = 2\,(a + b)\,c$.

The method of procedure then, is to find the largest root contained in 5270. The first three terms of the above formula, viz. $a^2 + 2\,a\,b + b^2$, show, that this is to be found by the method given above for finding a root consisting of two figures.

```
              52,70 (72
              49
              ─────
               37,0 (14
 72 × 72 =    51,84
              ─────
                 86
```

The root is 72, and the remainder is 86. Annex to this the two figures rejected above, and it becomes 8676. This contains $2\,(a + b)\,c + c^2$; that is,

$$2 \times 720 \times c + c^2.$$

If 8676 be divided by $2 \times 720 = 1440$, the quotient will be either c or a number larger by 1 or 2. The zero on the right of 1440, and the right hand figure in the dividend may be omitted without affecting the quotient. The quotient is 6. Put 6 into the root and raise the whole to the second power.

$$726 \times 726 = 527076$$

Operation.

52,70,76 (726 = root.
49

1st. dividend 37,0 (14 = 1st divisor.
72 × 72 = 51,84

2d dividend = 867,6 (144 = 2d divisor.
726 × 726 = 527,076.

There is, however, a method, which will save considerable labour in multiplying.

In the last example, for instance, having found the second figure of the root 2, instead of raising the whole 72 to the second power, we may abridge it very much by observing, that the second power of the 70, answering to a^2 in the formula, has already been found and subtracted; therefore it only remains to find $2\,a\,b + b^2$, and subtract it also. But the 140 is $2\,a$, and the figure 2 found for the root answers to b; therefore if we add 2 to 140, it becomes $142 = 2\,a + b$. If this be now multiplied by 2 or b, it becomes

$$2 \times 142 = 284 = 2\,a\,b + b^2.$$

This completes the second power of 72, which, subtracted from 370, leaves 86 as before.

Prepare as before, and find the third figure of the root. Observe that the 2d power of 720 or $a^2 + 2\,a\,b + b^2$ has already been found and subtracted; it only remains to find the other parts, viz. $2\,(a + b)\,c + c^2$. The divisor 1440 answers to $2\,(a + b)$. Add 6, the figure of the root just found, to this, and it becomes 1446, answering to $2\,(a + b) + c$. If this be multiplied by 6, it becomes $1446 \times 6 = 8676 = 2\,(a + b)\,c + c^2$. This completes the second power of 726, which, subtracted from 8676, the number remaining in the work, leaves nothing.

Operation.

	52,70,76, (726 root.		
	49		
1st dividend	370	14	1st divisor.
	284	142	1st multiplicand.
2d dividend	8676	144	2d divisor
	8676	1446	2d multiplicand.
	00		

The same principle will apply when the root consists of any number of figures whatever.

What is the root of 533837732164?

In the first place I observe that the second power of the tens can have no significant figure below hundreds, therefore the two right hand figures may be rejected for the present. Also the second power of the hundreds can have no significant figure below tens of thousands, therefore the next two may be rejected. For a similar reason the next two may be rejected. In this manner they may all be rejected two by two until only one or two remain. Begin by finding the root of these, and proceed as above.

Operation.

53,38,37,73,21,64 (730642
49

43,8 (143
42 9

93,7 (1460

9377,3 (14606
8763 6

613 72,1 (146124
584 49 6

29 22 564 (1461282
29 22 564.

After separating the figures two by two, as explained above, I find the greatest second power in the left hand division. It is 49, the root of which is 7. I subtract 49 from 53, and bring down the next two figures, which makes 438. Now considering the 7 as tens, I proceed as if I were finding the root of 5338; that is, I double the 7, which makes 14 for a divisor, and see how many times it is contained in 43, rejecting the 8 on the right. I find 3 times. I write 3 in the root at the right of 7, and also at the right of 14. I multiply 143 by 3, and subtract the product from 438. I then bring down the next two figures, which make 937. I double 73, or, which is the same thing, I double the 3 in 143; for the 7 was doubled to find 14. This gives 146 for a divisor. I seek how many times 146 is contained in 93, rejecting the 7 on the right, as before. I find it is not contained at all. I write zero in the root, and also at the right of 146. I then bring down the next two figures. I seek how many times 1460 is contained in 9377, rejecting the 3 on the right. I find 6 times. I write 6 in the root, and at the right of 1460, and multiply 14606 by 6, and subtract the product from 93773. I then bring down the next two figures, and double the right hand figure of the last multiplicand, and proceed as before; and so on, till all the figures are brought down. The doubling of the right hand figure of the last multiplicand, is always equivalent to doubling the root as far as it is found.

From the above examples, we derive the following rule for extracting the second root.

1st. *Beginning at the right, separate the number into parts of two figures each. The left hand part may consist of one or two figures.*

2nd. *Find the greatest second power in the left hand part, and write its root as a quotient in division. Subtract the second power from the left hand part.*

3d. *Bring down the two next figures at the right of the remainder. Double the root already found for a divisor. See how many times the divisor is contained in the dividend rejecting the right hand figure. Write the result in the root, at the right of the figure previously found, and also at the right of the divisor.*

4th. *Multiply the divisor, thus augmented, by the last figure of the root, and subtract the product from the whole dividend.*

5th. *Bring down the next two figures as before, to form a new dividend, and double the root already found, for a divisor, and proceed as before. The root will be doubled, if the right hand figure of the last divisor be doubled.*

If it happens that the divisor is not contained in the dividend when the right hand figure is rejected, a zero must be written in the root, and also at the right of the divisor; and the next figures must be brought down, and then a new trial made.

If it happens that the figure annexed to the root is too small, it may be discovered as follows.

The second power of $a + 1$ is $a^2 + 2a + 1$.

That is, if we have the second power of any number, the second power of a number larger by 1, is found by multiplying the first number by 2, increasing the product by 1, and adding it to the power. For example, the second power of 10 is 100; the second power of 11 is $100 + 2 \times 10 + 1 = 121$. The second power of 12 is $121 + 2 \times 11 + 1 = 144$, &c.

If then the remainder, after subtraction, is equal to twice the root already found plus 1, or greater, the last figure of the root must be increased by 1.

In the last example, the first dividend was 43,8 and the divisor 14; the figure put in the root was 3, and the remainder was 9. If 2 instead of 3 had been put in the root, the remainder would have been 154, which is considerably larger than twice 72, and would have shown, that the figure should be 3 instead of 2.

There are many numbers, of which the root cannot be exactly assigned in whole or mixed numbers. Thus 2, 3, 5, 6, 7, have no assignable roots. That is, no number can be found, which, multiplied into itself, shall produce either of these numbers. This is the case with all whole numbers, which have not an exact root in whole numbers.

This may be proved, but the demonstration is so difficult, that few learners would comprehend it at this stage of their progress. The proof may be found in Lacroix's Algebra. The learner, however, may easily satisfy himself by trial. We shall soon find a method of approximating the roots of these numbers, sufficiently near for all purposes.

XXIX. *Extraction of the second Root of Fractions.*

Fractions are multiplied together by multiplying their numerators together, and their denominators together. Hence the second power of a fraction is found by multiplying the numerator into itself, and the denominator into itself; thus the second power of $\frac{3}{5}$ is $\frac{3}{5} \times \frac{3}{5} = \frac{9}{25}$. The second power of $\frac{a}{b}$ is $\frac{a}{b} \times \frac{a}{b} = \frac{a^2}{b^2}$. Hence the root of a fraction is found by extracting the root of the numerator, and of the denominator; thus the root of $\frac{49}{81}$ is $\frac{7}{9}$.

If either the numerator or denominator has no exact root, the root of the fraction cannot be found exactly. Thus the root of $\frac{68}{81}$ is between $\frac{8}{9}$ and $\frac{9}{9}$ or 1. It is nearest to $\frac{8}{9}$.

The denominator of a fraction may always be rendered a perfect second power, so that its root may be found; and for the numerator, the number which is nearest to the root must be taken. Suppose it is required to find the root of $\frac{3}{5}$. If both terms of the fraction be multiplied by 5, the value of the fraction will not be altered, and the denominator will be a perfect second power,

$$\frac{3}{5} = \frac{15}{25}.$$

The root is nearest $\frac{4}{5}$. This is exact, within less than $\frac{1}{5}$.

If it is necessary to have the root more exactly; after the fraction has been prepared by multiplying both its terms by the denominator, we may again multiply both its terms by some number that is a perfect second power. The larger this number, the more exact the result will generally be.

$$\frac{3}{5} = \frac{15}{25}.$$

If both terms be multiplied by 144, which is the second power of 12, it becomes $\frac{2160}{3600}$, the root of which is nearest to $\frac{46}{60}$. This is the true root within less than $\frac{1}{60}$.

We may approximate in this way the roots of whole numbers, whose roots cannot be exactly assigned.

If it is required to find the root of 2, we may change it to a fraction, whose denominator is a perfect second power.

$$2 = \frac{288}{144}.$$

The root of $\frac{288}{144}$ is nearest to $\frac{17}{12} = 1\frac{5}{12}$. This differs from the true root by a quantity less than $\frac{1}{12}$. If greater exactness is required, a number larger than 144 may be used.

1. What is the root of $\frac{81}{121}$? *Ans.* $\frac{9}{11}$.
2. What is the root of $\frac{144}{625}$?
3. What is the root of $13\frac{129}{400} = \frac{5329}{400}$?
4. What is the root of $28\frac{317}{529}$?
5. What is the approximate root of $\frac{5}{9}$?
6. What is the approximate root of $\frac{18}{25}$?
7. What is the approximate root of $3\frac{2}{7}$?
8. What is the approximate root of $17\frac{3}{13}$?
9. What is the approximate root of 3?
10. What is the approximate root of 7?
11. What is the approximate root of 417?

The most convenient numbers to multiply by, in order to approximate the root more nearly, are the second powers of 10, 100, 1000, &c., which are 100, 10000, 1000000, &c. By this means, the results will be in decimals.

To find the root of 2 for instance, first reduce it to hundredths.

$2 = \frac{200}{100}$, the approximate root of which is $\frac{14}{10} = 1.4$.

Again $2 = \frac{20000}{10000}$, the approximate root of which is $\frac{141}{100} = 1.41$.

Again, $2 = \frac{2000000}{1000000}$, the approximate root of which is $\frac{1414}{1000} = 1.414$.

In this way we may approximate the root with sufficient accuracy for every purpose. But we may observe, that at every approximation, two more zeros are annexed to the number. In fact, if one zero is annexed to the root, there must be two annexed to its power; for the second power of 10 is 100, that of 100 is 10000, &c.

This enables us to approximate the root by decimals, and we may annex the zeros as we proceed in the work, always annexing two zeros for each new figure to be found in the root, in the same manner as two figures are brought down in whole numbers.

The root of 2 then may be found as follows.

```
2 (1.41421, &c. root.
1
----
10,0 (24
 9 6
-------
    40,0 (281
    28 1
    --------
    11 90,0 (2824
    11 29 6
    ----------
        60 40,0 (28282
        56 56 4
        -----------
          3 83 60,0 (282841
          2 82 84 1
          -----------
           1 00 75 9
```

12. What is the approximate root of 28?

13. What is the approximate root of 243?

14. What is the approximate root of 27068?

15. What is the approximate root of $243\frac{3}{8}$?

$243\frac{3}{8} = 243.\frac{375}{1000} = \frac{243375}{1000} = \frac{243375000}{1000000}$, &c.

The approximate root of which is $\frac{15600}{1000} = 15.6$, &c.

But it is plain that this may be performed in the same manner as the above. For if the number 243375000 be prepared in the usual way, it stands thus ; 2,43,37,50,00. Now

$$\frac{243375000}{1000000} = 243 . 375000.$$

If we take this number and begin at the units and point towards the left, and then towards the right in the same manner, the number will be separated into the same parts, viz. 2,43.37,50,00. The root of this number may be extracted in the usual way, and continued to any number of decimal places by annexing zeros.

N. B. The decimal point must be placed in the root, before the first two decimals are used. Or the root must contain one half as many decimal places as the power, counting the zeros which are annexed.

16. What is the approximate root of 213.53? 14.[illegible]
17. What is the approximate root of $726\frac{3}{8}$? 2[illegible]
18. What is the approximate root of $17\frac{5}{11}$? 4 [illegible]
19. What is the approximate root of $3\frac{1}{15}$? [illegible]
20. What is the approximate root of $\frac{3}{5}$?
21. What is the approximate root of $\frac{4}{5}$?
22. What is the approximate root of $\frac{4}{257}$?
23. What is the approximate root of $\frac{1}{1153}$?

XXX. *Questions producing pure Equations of the Second Degree.*

1. A mercer bought a piece of silk for £16. 4s.; and the number of shillings which he paid per yard, was to the number of yards, as 4 to 9. How many yards did he buy, and what was the price of a yard?

Let $x =$ the number of shillings he paid per yard.

Then $\frac{9x}{4} =$ the number of yards.

The price of the whole will be $\frac{9x^2}{4} = 324$ shillings.

$$x^2 = 144$$
$$x = 12$$
$$\frac{9x}{4} = 27.$$

Ans. 27 yards, at 12s. per yard.

2. A detachment of an army was marching in regular column, with 5 men more in depth than in front; but upon the enemy coming in sight, the front was increased by 845 men; and by this movement the detachment was drawn up in 5 lines. Required the number of men.

Let $x =$ the number in front;
then $x + 5 =$ the number in depth;
$x^2 + 5x =$ the whole number of men.
Again $x + 845 =$ the number in front after the movement;
And $5x + 4225 =$ the whole number.

$$\begin{aligned} x^2 + 5x &= 5x + 4225 \\ x^2 &= 4225 \\ x &= 65 \end{aligned}$$

The number of men $= 5x + 4225 = 4550$.

3. A piece of land containing 160 square rods, is called an acre of land. If it were square, what would be the length of one of its sides?

Let $x =$ one side.

$$\begin{aligned} x^2 &= 160 \\ x &= 12649 + \end{aligned}$$

Ans. The side is 12.649 + rods. It cannot be found exactly, because 160 is not an exact 2d power.

This is exact within less than $\frac{1}{1000}$ of a rod. It might be carried to a greater degree of exactness if necessary.

4. What is the side of a square field, containing 17 acres?

5. There is a field 144 rods long and 81 rods wide; what would be the side of a square field, whose content is the same?

6. A man wishes to make a cistern that shall contain 100 gallons, or 23100 cubic inches, the bottom of which shall be square, and the height 3 feet. What must be the length of one side of the bottom?

7. A certain sum of money was divided every week among the resident members of a corporation. It happened one week that the number resident was the root of the number of dollars to be divided. Two men however coming into residence the week after, diminished the dividend of each of the former individuals $1\frac{1}{3}$ dollars. What was the sum to be divided?

Let $x =$ the number of dollars to be divided;

then $x^{\frac{1}{2}}$ = the number of men resident, and also the sum each received.

The root of x is properly expressed by the fractional index $\frac{1}{2}$. For it has been observed, that when the same letter is found in two quantities which are to be multiplied together, the multiplication is performed, as respects that letter, by adding the exponents. Thus $a \times a = a^{1+1} = a^2$; $x^2 \times x^3 = x^{2+3} = x^5$, &c. Applying the same rule; if $x^{\frac{1}{2}}$ represents a root or first power, the second power or $x^{\frac{1}{2}} \times x^{\frac{1}{2}} = x^{\frac{1}{2}+\frac{1}{2}} = x^1$ or x.

The second power of a letter is formed from the first by multiplying its exponent by 2, because that is the same as adding the exponent to itself. Thus $a^3 \times a^3 = a^{3+3} = a^{2\times 3} = a^6$. This furnishes us with a simple rule to find the root of a literal quantity; which is, to divide its exponent by 2.

Thus the root of a^2 is $a^{\frac{2}{2}} = a^1$; the root of $a^4 = a^{\frac{4}{2}} = a^2$; the root of a^6 is $a^{\frac{6}{2}} = a^3$, &c. By the same rule, the root of a^1 is $a^{\frac{1}{2}}$; the root of a^3 is $a^{\frac{3}{2}}$; the root of a^5 is $a^{\frac{5}{2}}$; the root of a^n is $a^{\frac{n}{2}}$, &c.

In the above example

x = the number of dollars to be divided;

and $x^{\frac{1}{2}}$ = the number of men resident;

and $\dfrac{x}{x^{\frac{1}{2}}} = \dfrac{x^{\frac{1}{2}}\,x^{\frac{1}{2}}}{x^{\frac{1}{2}}} = x^{\frac{1}{2}}$ = the number of dollars each received.

$x^{\frac{1}{2}} + 2$ = the number of men the succeeding week;

$\dfrac{x}{x^{\frac{1}{2}}+2}$ = the number of dollars each received the latter week;

Hence by the conditions

$$x^{\frac{1}{2}} - \frac{4}{3} = \frac{x}{x^{\frac{1}{2}}+2}$$

$$x^{\frac{1}{2}}\, x^{\frac{1}{2}} - \frac{4\,x^{\frac{1}{2}}}{3} + 2\,x^{\frac{1}{2}} - \frac{8}{3} = x$$

$$x - \frac{4\,x^{\frac{1}{2}}}{3} + 2\,x^{\frac{1}{2}} - \frac{8}{3} = x$$

$$-\frac{4\,x^{\frac{1}{2}}}{3} + 2\,x^{\frac{1}{2}} - \frac{8}{3} = 0$$

$$-\frac{4}{3}\,x^{\frac{1}{2}} + 2\,x^{\frac{1}{2}} = \frac{8}{3}$$

$$-4\,x^{\frac{1}{2}} + 6\,x^{\frac{1}{2}} = 8$$

$$2\,x^{\frac{1}{2}} = 8$$

$$x^{\frac{1}{2}} = 4$$

$$x^{\frac{1}{2}} \times x^{\frac{1}{2}} = x^{\frac{1}{2}+\frac{1}{2}} = x = 4 \times 4 = 16.$$

Ans. $16.

Instead of making $x =$ the number of dollars, we might make,

$x^2 =$ the number of dollars ;
then $x =$ the number of men resident, &c.
Then we have

$$x - \frac{4}{3} = \frac{x^2}{x+2}$$

$$x^2 - \frac{4\,x}{3} + 2\,x - \frac{8}{3} = x^2$$

$$x^2 - \frac{4\,x}{3} + 2\,x - x^2 = \frac{8}{3}$$

$$2\,x = 8$$

$$x = 4$$

$$x^2 = 16.$$

Ans. $16, as before.

8. Two men, A and B, lay out some money on speculation. A disposes of his bargain for £11, and gains as much per cent. as B lays out; B's gain is £36, and it appears that A gains four times as much per cent. as B. Required the capital of each.

9. There is a rectangular field containing 360 square rods, and whose length is to its breadth as 8 to 5. Required the length and breadth.

10. There are two square fields, the larger of which contains 13941 square rods more than the smaller, and the proportion of their sides is as 15 to 8. Required the sides.

11. There is a rectangular room, the sum of whose length and breadth is to their difference as 8 to 1; if the room were a square whose side is equal to the length, it would contain 128 square feet more than it would, if it were only equal to the breadth. Required the length and breadth of the room.

12. There is a rectangular field, whose length is to its breadth in the proportion of 6 to 5. A part of this, equal to $\frac{1}{6}$ of the whole, being *planted*, there remain for *ploughing* 625 square yards. What are the dimensions of the field?

13. A charitable person distributed a certain sum amongst some poor men and women, the number of whom were in the proportion of 4 to 5. Each man received one third as many shillings as there were persons relieved; and each woman received twice as many shillings as there were women more than men. The men received all together 18s. more than the women. How many were there of each?

14. A man purchased a field whose length was to the breadth as 8 to 5. The number of dollars paid per acre was equal to the number of rods in the length of the field; and the number of dollars given for the whole, was equal to 13 times the number of rods round the field. Required the length and breadth of the field.

15. There is a stack of hay whose length is to its breadth as 5 to 4, and whose height is to its breadth as 7 to 8. It is worth as many cents per cubic foot as it is feet in breadth; and the whole is worth, at that rate, 224 times as many cents as there are square feet on the bottom. Required the dimensions of the stack.

16. There is a field containing 108 square rods, and the sum of the length and breadth is equal to twice the difference. Required the length and breadth.

17. There are two numbers whose product is 144, and the quotient of the greater by the less is 16. What are the numbers?

XXXI. *Questions producing Pure Equations of the Third Degree.*

1. A number of boys set out to rob an orchard, each carrying as many bags as there were boys in all, and each bag capable of containing 8 times as many apples as there were boys. They filled their bags, and found the whole number of apples was 1000. How many boys were there?

Let $x =$ the number of boys;

then $x \times x = x^2 =$ the number of bags;

and $8x \times x^2 = 8x^3 =$ the number of apples.

By the conditions

$$8x^3 = 1000$$

$$x^3 = 125$$

$$\text{or} \qquad xxx = 125.$$

In this equation, the unknown quantity is raised to the third power; and on this account is called an equation of the *third degree.*

In order to find the value of x in this equation, it is necessary to find what number multiplied twice by itself will make 125. By a few trials we find that 5 is the number; for

$$5 \times 5 \times 5 = 125$$

therefore $x = 5.$ *Ans.* 5 boys.

2. Some gentlemen made an excursion; and every one took the same sum of money. Each gentleman had as many servants attending him as there were gentlemen; and the number of dollars which each had, was double the number of all

the servants; and the whole sum of money taken out was $1458. How many gentlemen were there?

Ans. 9 gentlemen.

3. A poulterer bought a certain number of fowls. The first year each fowl had a number of chickens equal to the original number of fowls. He then sold the old ones. The next year each of the young ones had a number of chickens equal to once and one half the number which he first bought. The whole number of chickens the second year was 768. What was the number of fowls purchased at first?

It appears that in equations of the third degree, as in those of the second degree, the power of the unknown quantity must first be separated from the known quantities, and made to stand alone in one member of the equation, by the same rules as the unknown quantity itself is separated in simple equations. When this is done, the first power or the root must be found, and the work is finished.

Extraction of the Third Root.

The third power of a quantity is easily found by multiplication, but to return from the power to the root, is not so easy. It must be done by trial, in a manner analogous to that employed for the root of the second power.

We shall hereafter have occasion to speak of the root of the fourth power, of the fifth power, &c. In order to distinguish them the more readily, we shall call the root of the second power, the *second root* of the quantity; that of the third power, the *third root*, that of the fourth power, the *fourth root*, &c. To preserve the analogy, we shall sometimes call the root of the first power, the *first root.*

N. B. The first power, and the first root, are the same thing, and the same as the quantity itself.

It always has been, and is still the practice of mathematicians, to call the second root the *square root*, and the third root the *cube root*, and sometimes, though not so universally, the fourth root the *bi-quadrate* root. But as these terms are unappropriate, they will not be used in this treatise.

When the root consists of but one figure, it must be found by trial. When the root consists of more than one place, it must still be found by trial, but rules may be made, which will reduce the number of trials to very few, as has been done above for the second root.

In order to find the rules for extracting the third root, it will be necessary to observe how the third power is formed from the first, when the first consists of several figures.

Let $a = 30$ and $b = 5$; then $a + b = 35$.

$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$. Art. XIII.

$$\begin{array}{rrr} a^3 = & 30 \times 30 \times 30 & = 27000 \\ 3a^2b = & 3 \times 30 \times 30 \times 5 & = 13500 \\ 3ab^2 = & 3 \times 30 \times 5 \times 5 & = 2250 \\ b^3 = & 5 \times 5 \times 5 & = 125 \\ \hline & & 42875 \end{array}$$

Hence it appears, that the third power of a number consisting of units and tens, contains the third power of the tens, plus three times the second power of the tens multiplied by the units, plus three times the tens multiplied by the second power of the units, plus the third power of the units.

Farther, the third power of 10, which is the smallest number with two places, is 1000, which consists of four places; and the third power of 100, is 1000000, which consists of seven places. Hence the third power of tens will never be less than 1000, nor so much as 1000000.

If, therefore, there are tens in the root, their power will not be found below the fourth place; and if the root consists of tens without units, there will be no significant figure below 1000.

To trace back again the number 42875, the root of the tens will be found in the 42000, and this must be found by trial.

$$30 \times 30 \times 30 = 27000, \text{ and } 40 \times 40 \times 40 = 64000.$$

The largest third power in 42000 is 27000, the root of which is 30. Now I subtract 27000 from 42875, and the remainder is 15875, which contains the product of three times the second power of the tens by the units, plus, &c. If it contained exactly three times the second power of the tens multiplied by the units, the units of the root would be found immediately by dividing this remainder by three times the second power of the tens; for $3\,a^2\,b$ divided by $3\,a^2$ gives b. As the other parts however will always be small in comparison with this, if we divide the remainder by three times the second power of the tens, we shall be able to judge very nearly what is the root, and the number of trials will be limited to very few.

$30 \times 30 = 900$, and $900 \times 3 = 2700$ and 15875 divided by 2700 gives 5. I now add the 5 to the root and it becomes 35. To see if this is right, I raise 35 to the third power. $35 \times 35 \times 35 = 42875$, therefore 35 is the true root.

4. What is the third root of 79507?

Operation.

$$\begin{array}{l} 79{,}507\ (40 + 3 = 43 \text{ root.} \\ \underline{64{,}000} \\ 15{,}507\ (40 \times 40 \times 3 = 4800 \text{ divisor.} \end{array}$$

$43 \times 43 \times 43 = 79{,}507$.

As the number consists of five places, the power of the tens must be sought in the 79000.

The greatest third power in 79000 is 64000, the root of which is 40. I subtract 64000 from 79507 and there remains 15507, which I divide by three times the second power of 40, viz. 4800, and obtain a quotient 3, which I add to 40. I raise 43 to the third power, and find that it gives 79507. If it produced a number larger or smaller, I should put a smaller or larger number in place of 3 and try it again.

5. What is the third root of 357911?
6. What is the third root of 5832?
7. What is the third root of 941192?
8. What is the third root of 34965783?

It was observed above, that the third power of 10 is 1000; the third power of 100 is 1000000; that of 1000 is 1000000000, &c. That is, the third power of a number consisting of one figure cannot exceed three places; that of a number consisting of two places cannot contain less than 4 places nor more than 6; that of 3 places cannot contain less than 7 nor more than 9 places, &c.

Hence we may know immediately of how many places the third root of any given number will consist, by beginning at the right and separating the number into parts of 3 places each. The left hand part will not always contain 3 places.

In the present instance, the number 34,965,783, thus divided consists of three parts, therefore the root will contain 3 places or figures.

In the formula $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$, if we consider a as representing the hundreds of the root, and b the tens and units, we observe that the third power consists of the third power of the hundreds, plus 3 times the second power of the hundreds, multiplied by the units and tens, &c.

Hence we shall find the hundreds of the root by finding the highest third power contained in the 34,000,000, and taking its root.

The largest third power is 27,000,000, the root of which is 300. Subtracting 27,000,000 from the whole sum, the remainder is 7,965,783. If this contained exactly $3\,a^2\,b$, that is, 3 times the second power of the hundreds by the tens and units, the other two figures of the root might be found immediately by division. As it is, it is evident, that it will enable us to judge very nearly what the next figure, or tens, of the root must be, and its correctness must be proved by trial.

$$300 \times 300 \times 3 = 270000.$$

7,965,783 divided by 270000 gives for the first figure of the quotient 2, which being the tens is 20. This added to the root already found makes 320.

If in the above formula, we consider a as representing the hundreds and tens instead of the hundreds; and b as representing the units; it shows us that the power contains the third power of the hundreds and tens, plus 3 times the second power of the hundreds and tens multiplied by the units, &c. In the present instance $a = 320$. If now we subtract the third power of 320 from the whole sum, viz. 34,965,783, and divide the remainder by 3 times the second power of 320, we shall find the other figure, or units, of the root. When we have raised 320 to the third power, we can ascertain whether the second figure, 2 is right.

320 × 320 × 320 = 32768000.

This subtracted from 34965783 leaves 2197783.

320 × 320 × 3 = 307200.

2197783 being divided by 307200 gives a quotient 7. This added to 320 gives 327 for the root.

327 × 327 × 327 = 34,965,783.

Therefore the result is correct.

If the root consists of four or more places, the same mode of reasoning may be pursued by making *a* first equal to the highest figure in the root, and *b* equal to all below, until the second figure of the root is obtained, and then making *a* equal to the two figures already obtained, and *b* equal to the rest, and so on.

The work may be considerably abridged by omitting the zeros in the work, and also the numbers under which they fall.

The work of the above example will stand thus.

		Root.	
	34,965,783	(300 + 20 + 7 = 327.	
	— 27,000,000	3d power of 300	
1st divid.	7,965,783	(270,000	1st divisor = 300 × 300 × 3
	— 32,768,000	3d power of 320	
2d divid.	2,197,783	(307,200	2d divisor = 320 × 320 × 3
	34,965,783 = 3d power of 327.		

The same without the zeros.

	34,965,783	(327	
3d power of 3	27		
1st dividend	7,9	(27	1st divisor = $3^2 \times 3$
3d power of 32	32 768		
2d dividend	2 197,7	(3072	2d divisor = $(32)^2 \times 3$
	34,965,783.		

As the third power can have no significant figure below 1000000, and as the third power of 300 and 3 have the same significant figures, I raise 3 to the 3d power and subtract it from 34, as if it stood alone. Then, to form the divisor, hundreds are multiplied by hundreds, therefore there can be no significant figure below 10000. And it being the tens of the root that are to be found, it is sufficient to bring down one figure of the next period to form the dividend.

Having found the second figure of the root, I raise 32 to the third power, and subtract it from 34,965, omitting the last period, because the third power of the tens can have no significant figure below 1000.

To form the second divisor I multiply the second power of 32 by 3. For the dividend, it is sufficient to bring down one figure of the last period to the right of the remainder, because the divisor, being tens, multiplied by tens, can have no significant figure below 100.

Note. The second power of the 32 was found in finding its third power.

If it happens that the divisor is not contained in the dividend, a zero must be put in the root, and then the next figure must be brought down to form the dividend.

Hence we obtain the following rule for finding the third root.

Prepare the number by beginning at the right and separating it into parts or periods of three figures each, putting a comma or point between. The left hand period may consist of one, two, or three figures.

Find the greatest third power in the left hand period, and write the root in the place of a quotient. Subtract the power from the period. To the remainder bring down the first figure of the next period for a dividend. Multiply the second power of the root already found by three, to form a divisor. See how many times the divisor is contained in the dividend, and write the result in the root. Raise the root, thus augmented, to the third power. If this is greater than the first two periods, diminish the quotient by one or more, until you obtain a third power, which may be subtracted from the first two periods. Perform the subtraction, and to the right of the remainder bring down the first figure of the next period to form a dividend and divide it by three times the second power of *the two figures of the root, and write the quotient in the root. Then raise the whole root so found, to the third power; and if it is not too large, subtract it from the first three periods; if it is too large, diminish the root as before. To the remainder bring down the first figure of the fourth period, and perform the same series of operations as before.*

If at any time it should happen that the dividend, prepared as above, does not contain the divisor, a zero must be placed in the root, and the next figure brought down to form the dividend.

We explained a method in the extraction of the second root, more expeditious than to raise the root to the second power every time a new figure is obtained in the root. A similar method may be found for the third root, though it is rather difficult to be remembered.

Let $a = 30$ and $b = 7$; then

$$(a + b)^3 = (37)^3 = a^3 + 3\,a^2\,b + 3\,a\,b^2 + b^3 = 50653$$

To find the third root of 50653, find the first figure of the root as explained above. Then form the divisor as above, and find the second figure of the root. Then instead of raising the whole to the third power, it may be completed from the work already done. The third power of the first figure being found and subtracted, the remaining part is

$$3\,a^2\,b + 3\,a\,b^2 + b^3 = b\,(3\,a^2 + 3\,a\,b + b^2).$$

But the $3\,a^2$ has already been found for the divisor.

We must now find $3\,a\,b$ and b^2; add all together, and multiply the sum by b, and the third power will be completed.

Operation.

$$\begin{array}{rcrl}
3\,a^2 = & 3 \times 30 \times 30 = & 2700 & 50{,}6\,53\ (30 + 7 = 37. \\
3\,a\,b = & 30 \times 7 \times 3 = & 630 & \underline{27} \\
b^2 = & 7 \times 7 \quad = & \underline{49} & 23\,6{,}53\ (2700 = 3\,a^2, \\
 & & 7 \times 3379 = & 23\,6{,}53
\end{array}$$

9. What is the third root of 34,965,783?

We have seen above, that when the root is to consist of several figures, the same course is to be pursued as when it consists of only two.

Operation.

$3a^2 = 270000$	34,965,783 (300 + 20 + 7 = 327.
$3ab = 18000$	27 . . . . . .
$b^2 = 400$	
	79,65 (2700 1st divisor.
288400	57 68
$20 = b$	
	21 977,83 (307200 2d divisor..
5768000	21 977 83
	. . . . . .

$3(a^2 + 2ab + b^2) =$

$3a^2 + 2 \times 3ab + 3b^2$

$3a^2 = 270000$

$2 \times 3ab = 36000$

$3b^2 = 1200$

2d divisor $307200 = 3 \times 320 \times 320$

$3a^2 = 307200$

$3ab = 6720$

$b^2 = 49$

311969

$b = 7$

2197783

Examples.

10. What is the third root of 185193 ?
11. What is the third root of 8365427 ?
12. What is the third root of 77308776 ?

13. What is the third root of 1990865512?
14. What is the third root of 513,345,176,343?
15 What is the third root of 217,125,148,004,864?

XXXII. The third power of a fraction is found by raising both numerator and denominator to the third power. Thus the third power of $\frac{3}{5}$ is $\frac{3}{5} \times \frac{3}{5} \times \frac{3}{5} = \frac{27}{125}$.

Hence the third root of a fraction is found by finding the third root of both numerator and denominator. The third of $\frac{64}{343} = \frac{4}{7}$.

Examples.

1. What is the third root of $\frac{216}{343}$?
2. What is the third root of $\frac{27}{1000}$?
3. What is the third root of $3\frac{557}{729} = \frac{2744}{729}$?
4. What is the third root of $30\frac{41306}{50653}$?
5. What is the third root of $\frac{25}{8}$?

It was remarked with regard to the second root that, when a whole number has not an exact root in whole numbers, its root cannot be exactly found, for no fractional quantity multiplied by itself can produce a whole number. The same is true with regard to all roots, and for the same reason.

Hence the third root of $\frac{25}{8}$ cannot be found exactly because the numerator has no exact third root. The root of the denominator is 2, that of the numerator is between 2 and 3, nearest to 3. The approximate root is $\frac{3}{2}$ or $1\frac{1}{2}$.

6. What is the third root of $\frac{3}{7}$?

In this, neither the numerator nor the denominator is a perfect third power; but the denominator may be rendered a perfect third power, without altering the value of the fraction, by multiplying both terms of the fraction by 49, the second power of the denominator.

$$\frac{3 \times 49}{7 \times 49} = \frac{147}{343}.$$

The root of this is between $\frac{5}{7}$ and $\frac{6}{7}$, nearest to the former.

It is evident that the denominator of any fraction may be rendered a perfect third power, by multiplying both its terms by the second power of the denominator. The third root of a whole number which is not a perfect third power, may be approximated by converting the number into a fraction, whose denominator is a perfect third power.

What is the third root of 5?

We may find this root exact within less than $\frac{1}{12}$ of a unit, by converting it into a fraction, whose denominator is the third power of 12.

$$(12)^3 = 1728 \qquad 5 = \tfrac{8640}{1728}.$$

The root of $\frac{8640}{1728}$ is between $\frac{20}{12}$ and $\frac{21}{12}$; nearest the latter.

The most convenient numbers to multiply by, are the third powers of 10, 100, 1000, &c. in which case, the fractional part of the root will be expressed in decimals, in the same manner as was shown for the second root. The multiplication may be performed at each step of the work. For each decimal to be obtained in the root, three zeros must be annexed to the number, because the third power of 10 is 1000, that of 100, 1000000, &c.

7. The third root of 5 will be found by this method as follows.

	5.000,000,000	(1.709 +
3d power of 1	1	
1st dividend =	4 0	(3 1st divisor.
3d power of 1.7	4.913	
2d dividend =	870	(867 2d do. = 3 × $(17)^2$
3d do.	8700	(867 3d do.
3d power 1.709 =	4.991,443,829	
remainder	.008 556 171.	

The 3d root of 5 is 1.709, within less than $\frac{1}{1000}$ of a unit. We might approximate much nearer if necessary. The other method explained in the last article may be used if preferred.

8. What is the third root of $17\frac{3}{4}$?

The fractional part of this number must first be changed to a decimal.

$$17\tfrac{3}{4} = 17.75 = \tfrac{17750}{1000} = 17.750.$$

Hence it appears, that to prepare a number containing decimals, it is necessary that for every decimal place in the root, there should be three decimal places in the power. Therefore we must begin at the place of units, and separate the number both to the right and left into periods of three figures each. If these do not come out even in the decimals, they must be supplied by annexing zeros to the right.

9. What is the approximate third root of 25732.75?

10. What is the approximate third root of 23.1762?

11. What is the approximate third root of $12\frac{3}{5}$?

12. What is the approximate third root of $1\frac{17}{23}$?

13. What is the approximate third root of $\frac{13}{18}$?

14. What is the approximate third root of $\frac{2}{17}$?

XXXIII. *Questions producing Pure Equations of the Third Degree.*

1. A man wishes to make a cellar, that shall contain 31104 cubic feet; and in such a form, that the breadth shall be twice the depth, and the length $1\frac{1}{3}$ the breadth. What must be the length, breadth, and depth?

Let the depth $= x$,

the breadth $= 2x$,

and the length $= \frac{8x}{3}$.

The whole content will be

$$x \times 2x \times \frac{8x}{3} = 31104$$

$$\frac{16x^3}{3} = 31104$$

$$16x^3 = 93312$$
$$x^3 = 5832$$
$$x = 18 = \text{depth}$$
$$2x = 36 = \text{breadth}$$
$$\frac{8x}{3} = 48 = \text{length.}$$

2. There are two men whose ages are to each other as 5 to 4, and the sum of the third powers of their ages is 137781. What are their ages?

Let $x =$ the age of the elder

then $\frac{4x}{5} =$ the age of the younger.

$$x^3 + \frac{64x^3}{125} = 137{,}781$$
$$x^3 = 91{,}125$$
$$x = 45$$
$$\frac{4x}{5} = 36.$$

Ans. Elder 45 years, and younger 36.

3. A man wishes to make a cubical cistern that shall contain 100 gallons. What must be the length of one of its sides?

4. A bushel is 2150⅖ cubic inches. What must be the size of a cubical box to hold 1 bushel?

5. What must be the size of a cubical box to hold 2 bushels?

6. What must be the size of a cubical box to hold 8 bushels?

7. Find two numbers, such that the second power of the greater multiplied by the less may be equal to 448; and the second power of the less multiplied by the greater, may be 392?

8. A man wishes to make a cistern which shall hold 500 gallons, in such a form that the length shall be to the breadth as 5 to 4, and the depth to the length as 2 to 5. Required the length, breadth, and depth.

Note. The wine gallon is 231 cubic inches.

9. A man wishes to make a box which shall hold 40 bushels, in such form that the length shall be to the breadth as 4 to 3, and the depth to the breadth as 2 to 3. Required the length, breadth, and depth?

10. A man bought a piece of land for house lots, the breadth of which was to its length as 3 to 28; and he gave as many dollars per square rod, as there were rods in the length of the piece. The whole price was $63,504. Required the length and breadth.

11. A man agreed to sell a stack of hay for 10 times as many dollars as there were feet in the length of one of the longer sides. On measuring it, the length was to the breadth as 6 to 5, and the breadth and height were equal. Moreover it was found that it came to as many cents per cubic foot as there were feet in the breadth. Required the dimensions of the stack.

XXXIV. *Affected Equations of the Second Degree.*

When an equation of the second degree consists only of terms which contain the second power of the unknown quantity, and of terms entirely known, they may be solved as above. But an equation of the second power, in order to be complete, must contain both the first and second powers of the unknown quantity, and also one term consisting entirely of known quantities. These are sometimes called *affected equations*.

1. There is a field in the form of a rectangular parallelogram, whose length exceeds its breadth by 16 yards, and it contains 960 square yards. Required the length and breadth.

Let $x =$ the breadth;

then $x + 16 =$ the length;

and $x^2 + 16\,x =$ the number of square yards.

Hence $x^2 + 16\,x = 960$.

In order to solve this equation, it is necessary to make the first member a perfect second power.

Observe that the second power of the binomial $x + a$, is $x^2, + 2ax + a^2$, which consists of three terms.

Now if we compare this with the first member $x^2 + 16x$, we find

$$x^2 = x^2$$

$$2ax = 16x$$

which gives $2a = 16$

and $a = 8$

$$a^2 = 64$$

$$(x + 8)\ (x + 8) = x^2 + 16x + 64.$$

Hence, if to $x^2 + 16x$ we add 64, which is the second power of one half of 16, the first member will be a perfect second power, but it will be necessary to add the same quantity to the second member, in order to preserve the equality. The equation then becomes

$$x^2 + 16x + 64 = 960 + 64 = 1024.$$

Taking the root of both members

$$x + 8 = \pm (1024)^{\frac{1}{2}} = 32.$$

By transposition $x = -8 \pm 32.$

It has been already remarked that the 2nd root of every positive quantity, may be either positive or negative, because $-a \times -a = +a^2$ as well as $+a \times +a = +a^2$. The double sign ± is read plus or minus.

In the preceding examples, the conditions of the question have always determined which was to be used. But, in the present instance, the work not being completed when the root is taken, we must give it both signs, and when the values of x are found for both signs, the conditions will finally show which is to be used.

$$x + 8 = \pm 32.$$

If we use the sign $+$, we have

$$x = 24$$

and $$x + 16 = 40.$$

This gives the length 40 yards and the breadth 24. These numbers answer the conditions of the question.

If we use the sign $-$, we have

$$x = -40$$

$$x + 16 = -24.$$

These numbers will not satisfy the conditions of the question, but they will answer the conditions of the equation, as will be seen by putting them into the first equation.

$$-40 \times -40 + 16 \times -40 = 960.$$

2. A certain company at a tavern had a reckoning of 143 shillings to pay; but 4 of the company being so ungenerous as to slip away without paying, the rest were obliged to pay 1 shilling apiece more than they would have done, if all had paid. What was the whole number of persons?

Let $x =$ the number of persons at first;

then $x - 4 =$ the number after 4 have departed;

$\frac{143}{x} =$ the number of shillings each should have paid;

and $\frac{143}{x-4} =$ the number of shillings actually paid by each.

By the conditions

$$\frac{143}{x} + 1 = \frac{143}{x-4}.$$

Clearing of fractions

$$143\,x + x^2 - 572 - 4\,x = 143\,x$$

By transposition

$$x^2 - 4\,x = 572.$$

This equation is similar to the last, except in this, the second term of the first member has the sign $-$.

Here we must observe that the second power of the binomial $x - a$, is $x^2 - 2ax + a^2$, the same as that of $x + a$ with the exception of the sign of the second term.

In this equation, as before, we find two terms of the second power of a binomial; if we can find the other term we can easily solve the question.

It may be found as follows,

$$x^2 = x^2$$

$$2ax = -4x$$

$$2a = -4$$

which gives $$a = -2$$

and $$a^2 = 4$$

Adding 4 to both members of the equation it becomes

$$x^2 - 4x + 4 = 572 + 4 = 576.$$

Since -2 in this corresponds to a, the root of the first member is $x - 2$. In fact, $(x - 2)^2 = x - 4x + 4$. The root of 576 is 24.

Hence

$$x - 2 = \pm 24$$

$$x = 2 \pm 24.$$

The two values of x are 26 and -22. The former only answers the conditions of the question.

Proof. If the whole number, 26, had paid their shares, each would have paid $\frac{143}{26} = 5\frac{1}{2}$ shillings. But 22 only paid, consequently each paid $\frac{143}{22} = 6\frac{1}{2}$ shillings.

3. There are two numbers, whose difference is 9, and whose sum multiplied by the greater produces 266. What are those numbers?

Let $x =$ the greater;

then $x - 9 =$ the less,

$2x - 9 =$ their sum.

By the conditions

$$x(2x - 9) = 266$$

$$2x^2 - 9x = 266$$

$$x^2 - \frac{9x}{2} = 133.$$

If we use the general formula as before, we have

$$x^2 = x^2$$

$$2ax = -\frac{9x}{2}$$

$$2a = -\frac{9}{2}$$

$$a = -\frac{9}{4}.$$

$$a^2 = \frac{81}{16}.$$

Completing the second power, the equation becomes

$$x^2 - \frac{9x}{2} + \frac{81}{16} = 133 + \frac{81}{16} = \frac{2209}{16}.$$

Taking the root of both members

$$x - \frac{9}{4} = \pm\frac{47}{4}$$

$$x = \frac{9}{4} \pm \frac{47}{4}$$

which gives $x = \frac{56}{4} = 14$

and $x = -\frac{38}{4} = -9\frac{1}{2}$

$$x - 9 = 5$$

also $x - 9 = -18\frac{1}{2}$

Both values will answer the conditions of the question; for

$$14 + 5 = 19$$

and $$19 \times 14 = 266$$

also $$-9\tfrac{1}{2} + (-18\tfrac{1}{2}) = -28$$

and $$-28 \times -9\tfrac{1}{2} = 266.$$

In all the above examples, after the question was put into equation, the first thing done, was to reduce all the terms containing x^2 to one term, and those containing x into another, and to place them in one member of the equation, and to collect all the terms consisting entirely of known quantities into the other. This must always be done. Moreover x^2 must have the sign $+$ and its coefficient must be 1. The equation will then be in the following form.

$$x^2 + p\,x = q.$$

p and q being any known quantities and either positive or negative.

Every equation, however complicated, consisting of terms which contain x^2, and x, and known quantities may be reduced to this form.

Let the equation be

$$7 - \frac{3\,x}{5} = \frac{15 - x^2}{4\,x - 2}$$

Clearing of fractions it becomes

$$140\,x - 12\,x^2 - 70 + 6\,x = 75 - 5\,x^2.$$

Transposing and uniting terms

$$146\,x - 7\,x^2 = 145$$

Changing all the signs in both members

$$7\,x^2 - 146\,x = -145$$

Dividing by 7 (the coefficient of x^2)

$$x^2 - \frac{146\,x}{7} = -\frac{145}{7}$$

Here $p = -\frac{146}{7}$ and $q = -\frac{145}{7}$

To solve the equation

$$x^2 + p\,x = q.$$

We consider x^2 and $p\,x$ as two terms of the second power of the binomial $x + a$ in which

$$2\,a\,x = p\,x$$

$$2\,a = p$$

$$a = \frac{p}{2}$$

$$a^2 = \frac{p^2}{4}$$

Hence the binomial $x + a$ is equal to $x + \frac{p}{2}$, and the third term of the second power is $\frac{p^2}{4}$. In fact

$$\left(x + \frac{p}{2}\right)\left(x + \frac{p}{2}\right) = x^2 + p\,x + \frac{p^2}{4}.$$

Therefore the first member of the above equation may be rendered a complete second power, of which $x + \frac{p}{2}$ is the root, by adding to it $\frac{p^2}{4}$. The same quantity must be added to the second member, to preserve the equality.

The equation then becomes

$$x^2 + p\,x + \frac{p^2}{4} = q + \frac{p^2}{4}.$$

Taking the root of both members

$$x + \frac{p}{2} = \pm\left(q + \frac{p^2}{4}\right)^{\frac{1}{2}}$$

$$x = -\frac{p}{2} \pm \left(q + \frac{p^2}{4}\right)^{\frac{1}{2}}$$

From the above observations we derive the following general rule for the solution of equations which contain the first and second powers of the unknown quantity.

1st. *Prepare the equation, by collecting all the terms containing the first and second powers of the unknown quantity into the first member, and all the terms consisting entirely of known quantities into the other member. Unite all the terms containing the second power into one term, and all containing the first power into another. If the sign before the term containing the second power of the unknown quantity be not positive, make it so by changing all the signs of both members. If the coefficient of this term is not* 1, *make it so by dividing all the terms by its coefficient.*

2d. *Make the first member a complete second power. This is done by adding to both members the second power of half the coefficient of* x *(or of the first power of the unknown quantity.)*

3d. *Take the root of both members.*

The root of the first member will be a binomial, the first term of which will be the unknown quantity, and the second will be half the coefficient of x *as found above. The root of the second member must have the double sign* ±.

4th. *Transpose the term consisting of known quantities from the first to the second member, and the value of* x *will be found.*

4. A and B sold 130 ells of silk (of which 40 ells were A's and 90 B's) for 42 crowns. Now A sold for a crown one third of an ell more than B did. How many ells did each sell for a crown?

Let $x =$ the number of ells B sold for a crown; then $x + \frac{1}{3} =$ the number A sold for a crown;

$$\frac{90}{x} = \text{the price of 90 ells};$$

$$\frac{40}{x+\frac{1}{3}} = \text{the price of 40 ells.}$$

$$\frac{90}{x} + \frac{40}{x+\frac{1}{3}} = 42$$

$$90 + \frac{40\,x}{x + \frac{1}{3}} = 42\,x$$

$$90\,x + 30 + 40\,x = 42\,x^2 + 14\,x$$

$$116\,x - 42\,x^2 = -30$$

Changing signs $\quad 42\,x^2 - 116\,x = 30$

Dividing by 42 $\quad x^2 - \frac{116\,x}{42} = \frac{30}{42}$

Reducing fractions $\quad x^2 - \frac{58\,x}{21} = \frac{5}{7}.$

To complete the second power of the first member, take one half of $-\frac{58}{21}$, which is $-\frac{29}{21}$, and add its second power to both members.

$$x^2 - \frac{58\,x}{21} + \frac{841}{\overline{21)}^2} = \frac{5}{7} + \frac{841}{\overline{21)}^2} = \frac{315}{\overline{21)}^2} + \frac{841}{\overline{21)}^2} = \frac{1156}{\overline{21)}^2}$$

Taking the root of both members,

$$x - \frac{29}{21} = \pm \frac{34}{21}$$

$$x = \frac{29}{21} \pm \frac{34}{21}$$

Which give $\quad x = \frac{63}{21} = 3$

and $\quad x = -\frac{5}{21}$

The first value only will answer the conditions.

Ans. B sold 3 ells for a crown, and A $3\frac{1}{3}$.

The learner may observe, that in raising $\frac{29}{21}$ to the second power, I multiplied the numerator into itself, but expressed the power of the denominator by an exponent. This saved some work in this example. It may always be done when the number in the right hand member can be reduced to a fraction with the same denominator as the number added. In this case $\frac{5}{7}$ could be reduced to 21ths. The $\frac{5}{7}$ was reduced thus

$$\frac{5 \times 3}{7 \times 3} = \frac{15 \times 21}{21 \times 21} = \frac{315}{\overline{21})^2}$$

When the second member is a whole number, it can be reduced to a fraction with any denominator; consequently this form may be used.

5. A man bought a certain number of sheep for 80 dollars; if he had bought 4 more for the same money, they would have come to him 1 dollar apiece cheaper. What was the number of sheep?

6. A merchant sold a quantity of brandy for £39 and gained as much per cent. as the brandy cost him. How much did it cost him?

Let $x =$ the cost.

then $\frac{x}{100} =$ the rate per cent.

and $\frac{x^2}{100} =$ the gain.

also $39 - x =$ the gain.

7. Two persons, A and B, talking of their money, says A to B, if I had as many dollars as I have shillings, I should have as much money as you; but if I had as many shillings as their number multiplied by itself, I should have three times as much money as you, and 63 shillings over. How much money had each?

8. A colonel has a battalion of 1200 men, which he would draw up in a solid body of an oblong form, so that each rank may exceed each file by 59 men. What numbers must he place in rank and file?

9. A grazier bought as many sheep as cost him £60; out of which he reserved 15, and sold the remainder for £54, gaining 2 shillings a head by them. How many sheep did he buy, and what was the price of each?

10. A person bought two pieces of cloth of different sorts; of which the finer cost 4s. a yard more than the other. For the finer he paid £18; but for the coarser, which exceeded the finer in length by 2 yards, he paid only £16. How many

yards were there in each piece, and what was the price of each?

11. A labourer dug two trenches, one of which was 16 yards longer than the other, for $77.60; and the digging of each cost as many dimes per yard, as there were yards in length. What was the length of each?

12. There are two square buildings, that are paved with stones each a foot square. The side of one building exceeds that of the other by 12 feet, and both their pavements taken together contain 2120 stones. What are the lengths of them separately.

13. A man bought two sorts of linen for $\$13\frac{1}{6}$. A yard of the finer cost as many shillings as there were yards of the finer: Also 30 yards of the coarser, (which was the whole quantity,) were at such a price, that 7 yards cost as much as a yard of the finer. How many yards were there of the finer, and what was the value of each piece?

14. Two partners A and B gained £18 by trade. A's money was in trade 12 months, and he received for his principal and gain £26. Also B's money, which was £30, was in trade 16 months. What money did A put into trade?

15. The plate of a looking glass is 18 inches by 12, and is to be framed with a frame, all parts of which are of equal width, and the area of the frame is to be equal to that of the glass. Required the width of the frame.

16. A and B set out from two towns, which were distant 247 miles, and travelled the direct road till they met. A went 9 miles a day; and the number of days, at the end of which they met, was greater by 3 than the number of miles which B went in a day. How many miles did each go?

17. A set out from C towards D, and travelled 7 miles per day. After he had gone 32 miles, B set out from D towards C, and went every day $\frac{1}{19}$ of the whole journey; and after he had travelled as many days as he went miles in one day, he met A. What is the distance between the places C and D?

In this case both values will answer the conditions of the question.

18. A man had a field, the length of which exceeded the breadth by 5 rods. He gave 3 dollars a rod to have it fenced, which amounted to 1 dollar for every square rod in the field. What was the length and breadth, and what did he give for fencing it?

19. From two places at a distance of 320 miles, two persons, A and B, set out at the same time to meet each other. A travelled 8 miles a day more than B, and the number of days in which they met was equal to half the number of miles B went in a day. How many miles did each travel, and how far per day?

20. A man has a field 15 rods long and 12 rods wide, which he wishes to enlarge so that it may contain just twice as much; and that the length and breadth may be in the same proportion. How much must each be increased?

In this example, the root can be obtained only by approximation.

21. A square court yard has a rectangular gravel walk round it. The side of the court wants 2 yards of being 6 times the breadth of the gravel walk; and the number of square yards in the walk exceeds the number of yards in the periphery of the court by 164. Required the area of the court?

All equations of the second degree may be reduced to one of the following forms.

1. $x^2 + px = q$
2. $x^2 - px = q$
3. $x^2 + px = -q$
4. $x^2 - px = -q.$

After the equation has been brought to one of these forms, it may be solved by one of the following formulas, which are numbered to correspond to the equations from which they are derived.

1. $x = -\frac{p}{2} \pm \left(q + \frac{p^2}{4}\right)^{\frac{1}{2}}$

2. $x = +\frac{p}{2} \pm \left(q + \frac{p^2}{4}\right)^{\frac{1}{2}}$

3. $$x = -\frac{p}{2} \pm \left(\frac{p^2}{4} - q\right)^{\frac{1}{2}}$$

4. $$x = +\frac{p}{4} \pm \left(\frac{p^2}{4} - q\right)^{\frac{1}{2}}$$

The first equation and the first formula are sufficient for the whole, if p and q are supposed to be positive or negative quantities.

21. There are two numbers whose difference is $11\frac{3}{5}$, and whose product is equal to 4 times the larger minus 9. What are the numbers?

Let $x =$ the larger;

then $x - 11\frac{3}{5} =$ the smaller.

$$x^2 - 11\tfrac{3}{5}\,x = 4\,x - 9$$

$$x^2 - \tfrac{78x}{5} = -9.$$

This equation is in the form of $x^2 - p\,x = -q$, in which

$$p = \frac{78}{5},\ \frac{p}{2} = \frac{78}{10},\ \frac{p^2}{4} = \frac{6084}{100} \text{ and } q = 9.$$

$$x = \tfrac{78}{10} \pm \left(\tfrac{6084}{100} - 9\right)^{\frac{1}{2}} = \tfrac{78}{10} \pm \left(\tfrac{5184}{100}\right)^{\frac{1}{2}} = 7.8 \pm 7.2.$$

Or we may use the first formula, then

$$p = -\frac{78}{5},\ \frac{p}{2} = -\frac{78}{10},\ \frac{p^2}{4} = \frac{6084}{100}, \text{ and } q = -9$$

$$x = \tfrac{78}{10} \pm \left(\tfrac{6084}{100} - 9\right)^{\frac{1}{2}} = \tfrac{78}{10} \pm \left(\tfrac{5184}{100}\right)^{\frac{1}{2}} = 7.8 \pm 7.2.$$

Both values of x, being positive, will answer the conditions of the question.

Ans. By the first value the larger number is 15 and the smaller $3\frac{2}{5}$. By the second value of x, the larger is $\frac{3}{5}$, and the smaller -11.

Let the learner solve some of the preceding questions by the formula.

XXXV. We shall now demonstrate that every equation of the second degree, necessarily admits of two values for the unknown quantity, and only two.

Let us take the general equation.

$$x^2 + p\,x = q.$$

This, we have seen, may represent any equation whatever of the second degree, p and q being any known quantities and either positive or negative. If $p = 0$ the equation becomes

$$x^2 = q,$$

which is a pure equation or an equation with two terms.

If we make the first member of the equation $x^2 + p\,x = q$, a complete second power, by the above rules, it becomes

$$x^2 + p\,x + \frac{p^2}{4} = q + \frac{p^2}{4}$$

or $$\left(x + \frac{p}{2}\right)^2 = q + \frac{p^2}{4}.$$

Make $$m^2 = q + \frac{p^2}{4}$$

then $$m = \left(q + \frac{p^2}{4}\right)^{\frac{1}{2}}$$

Then we have $\left(x + \frac{p}{2}\right)^2 = m^2$

transposing m^2 $\left(x + \frac{p}{2}\right)^2 - m^2 = 0.$

The first member of this equation is the difference of two second powers, which, Art. XIII, is the same as the product of the sum and difference of the numbers.

The sum is $x + \frac{p}{2} + m$, and the difference is $x + \frac{p}{2} - m$,

and their product is

$$\left(x + \frac{p}{2} - m\right) \left(x + \frac{p}{2} + m\right) = 0.$$

In this equation, the first member consists of two factors, and the second is zero. Now the first member of the above equation will be equal to zero, if either of its factors is equal to zero. For if any number be multiplied by zero, the product is zero.

Making the first factor equal to zero,

$$x+\frac{p}{2}-m=0$$

gives $$x=-\frac{p}{2}+m.$$

Making $$x+\frac{p}{2}+m=0$$

gives $$x=-\frac{p}{2}-m.$$

Either of these values of x must answer the conditions of the equation.

N. B. Though either value answers the conditions separately, they cannot be introduced together, for being different; their product cannot be x^2.

Instead of m put its value, and the values of x become

$$x=-\frac{p}{2}+\left(\frac{p^2}{4}+q\right)^{\frac{1}{2}}$$

$$x=-\frac{p}{2}-\left(\frac{p^2}{4}+q\right)^{\frac{1}{2}}$$

which are the values we had obtained above. (This demonstration is essentially that of M. Bourdon.)

Discussion.

Let us take again the general equation.

$$x=-\frac{p}{2}\pm\left(q+\frac{p^2}{4}\right)^{\frac{1}{2}}.$$

Since the expression contains a radical quantity, that is, a quantity of which the root is to be found, in order to be able to find the value of it, we must be able to find the root either exactly or by approximation. Now there is one case in which it is impossible to find the root. It is when q is negative and greater than $\frac{p^2}{4}$. In which case the expression $q + \frac{p^2}{4}$ is negative; and it has been shown above, that it is impossible to find the root of a negative quantity. In all other cases the value of the equation may be found.

In all cases if q is positive, the first value will be positive, and answer directly to the conditions of the question proposed. For the radical $\left(q + \frac{p^2}{4}\right)^{\frac{1}{2}}$ is necessarily greater than $\frac{p}{2}$, because the root of $\frac{p^2}{4}$ alone is $\frac{p}{2}$; therefore the expression $-\frac{p}{2} \pm \left(q + \frac{p^2}{4}\right)^{\frac{1}{2}}$ is necessarily of the same sign as the radical.

The second value is for the same reason essentially negative, for both $\frac{p}{2}$ and $\left(q + \frac{p^2}{4}\right)^{\frac{1}{2}}$ are negative. This value, though it fulfils the conditions of the equation, does not answer the conditions of the question, from which the equation was derived; but it belongs to an analogous question, in which the x must be put in with the sign $-$ instead of $+$; thus $x^2 - p\,x = q$, which gives $x = \frac{p}{2} \pm \left(q + \frac{p^2}{4}\right)^{\frac{1}{2}}$, a value, which differs from the first only by the sign before $\frac{p}{2}$.

If q is actually negative, the equation becomes

$$x^2 \pm p\,x = -q,$$

and the values are

$$x = \mp \frac{p}{2} \pm \left(\frac{p^2}{4} - q\right)^{\frac{1}{2}}.$$

In order that it may be possible to find the root, q must be less than $\frac{p^2}{4}$. When this is the case, the two values are real.

Since $\left(\frac{p^2}{4}-q\right)^{\frac{1}{2}}$ is smaller than $\frac{p}{2}$, it follows that both values are negative if p is positive in the equation; that is, if $x^2+p\,x=-q$, which gives

$$x=-\frac{p}{2}\pm\left(\frac{p^2}{4}-q\right)^{\frac{1}{2}};$$

and both positive if p is negative in the equation, that is, $x^2-p\,x=-q$, which gives

$$x=\frac{p}{2}\pm\left(\frac{p^2}{4}-q\right)^{\frac{1}{2}}.$$

When both values are negative, neither of them answers directly to the conditions of the question; but if $-x$ be put into the original equation instead of x, the new equation will show what alteration is to be made in the enunciation of the question; and the same values will be found for x as before, with the exception of the signs.

If in this equation q is greater than $\frac{p^2}{4}$, the quantity $\left(\frac{p^2}{4}-q\right)^{\frac{1}{2}}$ becomes negative, and the extraction of the root cannot be performed. The values are then said to be *imaginary*.

1. It is required to find two numbers whose sum is p, and whose product is q.

Let x = one of the numbers,

then $p-x$ = the other.

$$x(p-x)=q$$

$$p\,x-x^2=q;$$

Changing signs $\quad x^2-p\,x=-q.$

This example presents the case above mentioned, in which p and q are both negative.

The value is

$$x = \frac{p}{2} \pm \left(\frac{p^2}{4} - q\right)^{\frac{1}{2}}.$$

Suppose $p = 15$ and $q = 54$.

$$x = \frac{15}{2} \pm \left(\frac{225}{4} - 54\right)^{\frac{1}{2}} = \frac{15}{2} \pm \left(\frac{225 - 216}{4}\right)^{\frac{1}{2}}$$

$$= \frac{15}{2} \pm \frac{3}{2}.$$

The values are 9 and 6, both positive, and both answer the conditions of the question. And these are the two numbers required, for $9 + 6 = 15$, $9 \times 6 = 54$. This ought to be so, for x in the equation represents either of the numbers indifferently. Indeed whichsoever x be put for, $p - x$ will represent the other; and $p\,x - x^2$ will be their product.

Again let $p = 16$ and $q = 72$.

$$x = \frac{16}{2} \pm \left(\frac{256}{4} - 72\right)^{\frac{1}{2}} = 8 \pm (-8)^{\frac{1}{2}}.$$

Here $(-8)^{\frac{1}{2}}$ is an imaginary quantity, therefore both values are imaginary.

In order to discover why we obtain this imaginary result, let us first find into what two parts a number must be divided, that the product of the two parts may be the greatest possible quantity. In the above example, p represents the sum of the two numbers or parts, let d represent their difference, then

$\frac{p}{2} + \frac{d}{2} =$ the greater, and $\frac{p}{2} - \frac{d}{2} =$ the less. Art. IX.

Their product is

$$\left(\frac{p}{2} + \frac{d}{2}\right)\left(\frac{p}{2} - \frac{d}{2}\right) = \frac{p^2}{4} - \frac{d^2}{4} \qquad \text{Art. XIII.}$$

The expression $\frac{p^2}{4} - \frac{d^2}{4}$ is evidently less than $\frac{p^2}{4}$ so long as d is greater than zero; but when $d = 0$, the expression becomes

$\frac{p^2}{4}$ which is the second power of $\frac{p}{2}$. Therefore the greatest possible product is when the two parts are equal.

In the above example $\frac{p}{2} = 8$, and $\frac{p^2}{4} = 64$. This is the greatest possible product that can be formed of two numbers whose sum is 16. It was therefore absurd to require the product to be 72; and the imaginary values of x arise from that absurdity.

2. It is required to find a number such, that if to its second power, 9 times itself be added, the sum will be equal to three times the number less 5.

$$x^2 + 9x = 3x - 5.$$

$$x^2 + 6x = -5.$$

This equation is in the form of $x^2 + px = -q$, which gives

$$x = -\frac{p}{2} \pm \left(\frac{p^2}{4} - q\right)^{\frac{1}{2}}$$

Putting in the values of p and q

$$x = -3 \pm (9-5)^{\frac{1}{2}} = -3 \pm 2.$$

The values are -1 and -5, both negative. Consequently neither value will answer the conditions of the question. This shows also that those conditions cannot be answered.

But if we change the sign of x in the equation, that is, put in $-x$ instead of x, it becomes

$$x^2 - 9x = -3x - 5.$$

Changing all the signs

$$9x - x^2 = 3x + 5.$$

This shows that the question should be expressed thus:

It is required to find a number, such, that if from 9 times itself, its second power be subtracted, the remainder will be equal to 3 times the number plus 5.

The values will both be positive in this, and both answer the conditions.

$$x^2 - 9x = -3x - 5$$

$$x^2 - 6x = -5$$

$$x = 3 \pm (9 - 5)^{\frac{1}{2}} = 3 \pm 2.$$

The values are 5 and 1 as before, but now both are positive, and both answer the conditions of the question.

3. There are two numbers whose sum is a, and the sum of whose second powers is b. It is required to find the numbers.

Examine the various cases which arise from giving different values to a and b. Also how the negative value is to be interpreted. Do the same with the following examples.

4. There are two numbers whose difference is a, and the sum of whose second powers is b. Required the numbers.

5. There are two numbers whose difference is a, and the difference of whose third powers is b. Required the numbers.

6. A man bought a number of sheep for a number a of dollars; and on counting them he found that if there had been a number b more of them, the price of each would have been less by a sum c. How many did he buy?

7. A grazier bought as many sheep as cost him a sum a, out of which he reserved a number b, and sold the remainder for a sum c, gaining a sum d per head by them. How many sheep did he buy, and what was the price of each?

8. A merchant sold a quantity of brandy for a sum a, and gained as much per cent. as the brandy cost him. What was the price of the brandy?

XXXVI. *Of Powers and Roots in General.*

Some explanation of powers both of numeral and literal quantities was given Art. X. The method of finding the roots of the second and third powers, that is, of finding the second and third roots of numeral quantities, has also been explained; and their application to the solution of equations. But it is frequently necessary to find the roots of other powers, as well as of the second and third, and of literal, as well as of numeral quantities. Preparatory to this, it is necessary to attend a little more particularly to the formation of powers.

The second power of a is $a \times a = a^2$.

The fifth power of a is $a \times a \times a \times a \times a = a^5$.

If a quantity as a is multiplied into itself until it enters m times as a factor, it is said to be raised to the mth power, and is expressed a^m. This is done by $m - 1$ multiplications; for one multiplication as $a \times a$ produces a^2 the second power, two multiplications produce the third power, &c.

We have seen above Art. X. that when the quantities to be multiplied are alike, the multiplication is performed by adding the exponents. By this principle it is easy to find any power of a quantity which is already a power. Thus

The second power of a^3 is $a^3 \times a^3 = a^{3+3} = a^6$.

The third power of a^2 is $a^2 \times a^2 \times a^2 = a^{2+2+2} = a^6$.

The second power of a^m is $a^m \times a^m = a^{m+m} = a^{2m}$.

The third power of a^m is $a^m \times a^m \times a^m = a^{m+m+m} = a^{3m}$.

The mth power of a^2 is $a^2 \times a^2 \times a^2 \times a^2 \times \ldots\ldots = a^{2+2+2+2+\ldots\ldots}$, until a^2 is taken m times as a factor, that is, until the exponent 2 has been taken m times. Hence it is expressed a^{2m}.

The nth power of a^m is $a^m \times a^m \times a^m \ldots\ldots = a^{m+m+m+\ldots}$ until m is taken n times, and the power is expressed a^{mn}.

N. B. The dots ..... in the two last examples are used to express the continuation of the multiplication or addition, because it cannot come to an end until m in the first case, and n in the second, receive a determinate value.

In looking over the above examples we observe;

1st. That the second power of a^3 is the same as the third power of a^2, and so of all others.

2. That in finding a power of a letter the exponent is added until it is taken as many times as there are units in the exponent of the required power. *Hence any quantity may be raised to any power by multiplying its exponent by the exponent of the power to which it is to be raised.*

The 5th power of a^3 is $a^{3\times5} = a^{15}$.

The 3d power of a^7 is $a^{7\times3} = a^{21}$, &c.

The power of a product is the same as the product of that power of all its factors.

The 2d power of $3\,a\,b$ is $3\,a\,b \times 3\,a\,b = 9\,a^2\,b^2$.

The 3d power of $2\,a^2\,b^3$ is $2\,a^2\,b^3 \times 2\,a^2\,b^3 \times 2\,a^2\,b^3 = 8\,a^6\,b^9$.

Hence, *when a quantity consists of several letters, it may be raised to any power by multiplying the exponents of each letter by the exponent of the power required; and if the quantity has a numeral coefficient, that must be raised to the power required.*

The powers of a fraction are found by raising both numerator and denominator to the power required; for that is equivalent to the continued multiplication of the fraction by itself.

1 What is the 5th power of $3\,a^2\,b^2\,m$?

2 What is the 3d power of $\frac{2\,a^5\,c^2}{5\,b^4\,d^3}$?

Powers of compound quantities are found like those of simple quantities, by the continued multiplication of the quantity into itself. The second power is found by multiplying the quantity once by itself. The third power is found by two multiplications, &c.

The powers of compound quantities are expressed by enclosing the quantities in a parenthesis, or by drawing a vinculum over them, and giving them the exponent of the power. The third power of $a + 2\,b - c$ is expressed $(a + 2\,b - c)^3$; or $\overline{a + 2\,b - c}^{\,3}$.

The powers are found by multiplication as follows:

$$\begin{array}{l}
a + 2b - c \\
a + 2b - c \\
\hline
a^2 + 2ab - ac \\
\qquad 2ab + 4b^2 - 2bc \\
\qquad\qquad - ac - 2bc + c^2 \\
\hline
a^2 + 4ab + 4b^2 - 2ac - 4bc + c^2 = (a + 2b - c)^2 \\
a + 2b - c \\
\hline
a^3 + 4a^2b + 4ab^2 - 2a^2c - 4abc + ac^2 \\
\qquad 2a^2b + 8ab^2 + 8b^3 - 4abc - 8b^2c + 2bc^2 \\
\qquad\qquad\qquad - a^2c - 4abc - 4b^2c + 2ac^2 + 4bc^2 - c^3 \\
\hline
a^3 + 6a^2b + 12ab^2 + 8b^3 - 3a^2c - 12abc - 12b^2c \\
\qquad\qquad\qquad + 3ac^2 + 6bc^2 - c^3 = (a + 2b - c)^3.
\end{array}$$

If the third power be multiplied by $a + 2b - c$, it will produce the fourth power.

3. What is the second power of $3c + 2d$?

4. What is the third power of $4a - bc$?

5. What is the fifth power of $a - b$?

6. What is the fourth power of $2a^2c - c^2$?

In practice it is generally more convenient to express the powers of compound quantities, than actually to find them by multiplication. And operations may frequently be more easily performed on them when they are only expressed.

$$(a + b)^3 \times (a + b)^2 = (a + b)^{3+2} = (a + b)^5$$
$$(3a - 5c)^4 \times (3a - 5c)^3 = (3a - 5c)^7.$$

That is, *when one power of a compound quantity is to be multiplied by any power of the same quantity, it may be expressed by adding the exponents, in the same manner as simple quantities.*

The 2d power of $(a+b)^3$ is $(a+b)^3 \times (a+b)^3$
$$= (a+b)^{3+3} = (a+b)^{3\times 2} = (a+b)^6.$$

The 3d power of $(2a-d)^4$ is
$$(2a-d)^{4+4+4} = (2a-d)^{4\times 3} = (2a-d)^{12}.$$

That is, *any quantity, which is already a power of a compound quantity, may be raised to any power by multiplying its exponent by the exponent of the power to which it is to be raised.*

7. Express the 2d power of $(3b-c)^4$.

8. Express the 3d power of $(a-c+2d)^5$.

9. Express the 7th power of $(2a^2-4c^3)^3$.

Division may also be performed by subtracting the exponents as in simple quantities.

$(3a-b)^5$ divided by $(3a-b)^3$ is
$$(3a-b)^{5-3} = (3a-b)^2$$

10. Divide $(7m+2c)^7$ by $(7m+2c)^3$.

If $(a+b)^2$ is to be multiplied by any quantity c, it may be expressed thus: $c\,(a+b)^2$. But in order to perform the operation, the 2d power of $a+b$ must first be found.

$$c\,(a+b)^2 = c\,(a^2+2ab+b^2) = a^2c+2abc+b^2c$$

If the operation were performed previously, a very erroneous result would be obtained; for $c\,(a+b)^2$ is very different from $(ac+bc)^2$. The value of the latter expression is $a^2c^2+2abc^2+b^2c^2$.

11. What is the value of $2\,(a+3b)^3$ developed as above?

12. What is the value of $3bc\,(2a-c)^2$?

13. What is the value of $(a+3c^2)\,(3a-2b)^2$?

14. What is the value of $(2a-b)^2\,(a^2+bc)^2$?

We have had occasion in the preceding pages to return from the second and third powers to their roots. We have shown how this can be done in numeral quantities; it remains to be shown how it may be effected in literal quantities. It is frequently necessary to find the roots of other powers as well as of the second and third.

The power of a literal quantity, we have just seen, is found by multiplying its exponent by the exponent of the power to which it is to be raised.

The second power of a^3 is $a^{3 \times 2} = a^6$; consequently the second root of a^6 is $a^{\frac{6}{2}} = a^3$.

The third power of a^m is a^{3m}; hence the third root of a^{3m} must be $a^{\frac{3m}{3}} = a^m$.

The second root of a^m, then must be $a^{\frac{m}{2}}$.

Proof. The second power of $a^{\frac{m}{2}}$ is $a^{\frac{2m}{2}} = a^m$.

In general, *the root of a literal quantity may be found by dividing its exponent by the number expressing the root; that is, by dividing by* 2 *for the second root, by* 3 *for the third root*, &c. This is the reverse of the method of finding powers.

It was shown above, that any power of a quantity consisting of several factors is the same as the product of the powers of the several factors. From this it follows, that any root of a quantity consisting of several factors is the same as the product of the roots of all the factors.

The third power of $a^2\ b\ c^3$ is $a^6\ b^3\ c^9$; the third root of $a^6\ b^3\ c^9$ must therefore be $a^2\ b\ c^3$.

Numeral coefficients are factors, and in finding powers they are raised to the power; consequently in finding roots, the root of the coefficient must be taken.

The 2nd root of $16\ a^4\ b^2$ is $4\ a^2\ b$.

Proof. $4\ a^2\ b \times 4\ a^2\ b = 16\ a^4\ b^2$.

When the exponent of a quantity is divisible by the number expressing the degree of the root, the root can be found exactly; but when it is not, the exponent of the root will be a fraction.

The second root of a^3 is $a^{\frac{3}{2}}$. The second root of a is $a^{\frac{1}{2}}$. The third root of a is $a^{\frac{1}{3}}$. The nth root of a is $a^{\frac{1}{n}}$. The nth root of a^m is $a^{\frac{m}{n}}$.

The root of a fraction is found by taking the root of its numerator and of its denominator. This is evident from the method of finding the powers of fractions.

The root of any quantity may be expressed by enclosing it in a parenthesis or drawing a vinculum over it, and writing a fractional exponent over it, expressive of the root. Thus

The 3d root of $8\,a^3\,b$ is expressed

$$(8\,a^3\,b)^{\frac{1}{3}} \text{ or } \overline{8\,a^3\,b}^{\frac{1}{3}}.$$

The root of a compound quantity may be expressed in the same way.

The 4th root of $a^2 + 5\,a\,b$ is expressed

$$(a^2 + 5\,a\,b)^{\frac{1}{4}} \text{ or } \overline{a^2 + 5\,a\,b}^{\frac{1}{4}}.$$

When a compound quantity has an exponent, its root may be found in the same manner as that of a simple quantity.

The 3d root of $(2\,b - a)^6$ is $(2\,b - a)^{\frac{6}{3}} = (2\,b - a)^2$.

With regard to the signs of roots it may be observed, that all even roots must have the double sign $\pm$; for since all even powers are necessarily positive, it is impossible to tell whether the power was derived from a positive or negative root, unless something in the conditions of the question shows it. An even root of a negative quantity is impossible. All odd roots will have the same sign as the power.

15. What is the second root of $9\,a^2\,b^6$?

16. What is the third root of $-\,125\,a^5\,b^6\,c$?

17. What is the fifth root of $32\,a^{10}\,x^m\,r$?

18. What is the third root of $\frac{a^3 b^3}{27 c^3 d^6}$?

19. What is the fourth root of $\frac{81 a^8 c^3}{b^2 m}$?

20. What is the second root of $(2m - x)^6$?

21. What is the 6th root of $(3a + x)^m$?

XXXVII. *Roots of Compound Quantities.*

When a compound quantity is a perfect power, its root may be found; and when it is not a perfect power, its root may be found by approximation, by a method similar to that employed for finding the roots of numeral quantities.

First we may observe, that no quantity consisting of only two terms can be a complete power; for the second power of a binomial consists of three terms; that of $a + x$, for example, is $a^2 + 2ax + x^2$. The quantity $a^2 + b^2$ is not a complete second power.

Let it be required to find the second root of

$$9 x^4 a^6 + 4 a^2 b^4 + 12 x^2 a^4 b^2.$$

The root of this will consist of at least two terms. The second power of the binomial $a + b$ is $a^2 + 2ab + b^2$. This shows that the quantity must be arranged according to the powers of some letter as in division, for the second power of either term of the root will produce the highest power of the letters in that term.

Arrange the above according to the powers of x.

$$9 x^4 a^6 + 12 x^2 a^4 b^2 + 4 a^2 b^4.$$

The formula $a^2 + 2ab + b^2$ shows that we should find the first term a of the root by taking the root of the first term; the same must be the case in the given example.

The root of $9 x^4 a^6$ is $3 x^2 a^3$. Write this in the place of a quotient, and subtract its second power. Then multiply $3 x^2 a^3$ by 2 for a divisor, answering to $2a$ of the formula.

$$
\begin{array}{lll}
9\,x^4\,a^6 + 12\,x^2\,a^4\,b^2 + 4\,a^2\,b^4 & & (3\,x^2\,a^3 + 2\,a\,b^2 \\
\underline{9\,x^4\,a^6} & & \\
\quad * \qquad 12\,x^2\,a^4\,b^2 + 4\,a^2\,b^4 & & (6\,x^2\,a^3 + 2\,a\,b^2 \\
\qquad\quad \underline{12\,x^2\,a^4\,b^2 + 4\,a^2\,b^4} & & \\
\qquad\qquad\quad * \qquad\quad * & &
\end{array}
$$

Divide the next term by the divisor. This gives $2\,a\,b^2$ for the next term of the root. Raise the whole root then to the second power and subtract it. Or, which is the same thing, since the second power of the first term has already been subtracted, write the quantity $2\,a\,b^2$ at the right of the divisor as well as in the root. Multiply the whole divisor as it then stands by the last term of the root. This produces the terms corresponding to $2\,a\,b + b^2, = b\,(2\,a + b)$ of the formula. This produces $12\,x^2\,a^4\,b^2 + 4\,a^2\,b^4$, which being subtracted, there is no remainder. Consequently the root is $3\,x^2\,a^3 + 2\,a\,b^2$ or $-3\,x^2\,a^3 - 2\,a\,b^2$. The second power of both is the same. If the double sign had been given to the first term of the root, the second would have had it also, and the positive and negative roots would have been obtained together.

Let it be required to find the 2d root of

$$36\,a^2\,m^4 - 60\,a\,b\,m^2 + 25\,b^2.$$

$$
\begin{array}{ll}
36\,a^2\,m^4 - 60\,a\,b\,m^2 + 25\,b^2 & (6\,a\,m^2 - 5\,b \\
\underline{36\,a^2\,m^4} & \\
\quad * \quad - 60\,a\,b\,m^2 + 25\,b^2 & (12\,a\,m^2 - 5\,b \\
\qquad\ \ \underline{- 60\,a\,b\,m^2 + 25\,b^2} & \\
\qquad\qquad\quad * \qquad\quad * &
\end{array}
$$

The process in this case is the same as in the last example. The second term of the root has the sign — in consequence of the term $60\,a\,b\,m^2$ of the dividend being affected with that sign. If the quantity had been arranged according to the powers of the letter b, thus, $25\,b^2 - 60\,a\,b\,m^2 + 36\,a^2\,m^4$, the root would have been $5\,b - 6\,a\,m^2$ instead of $6\,a\,m^2 - 5\,b$. Both roots are right, for the second powers of the two quantities are the same. The second power $a - b$ is the same as that of $b - a$. One is the positive and the other the negative root. If the double sign be given to the first term of the root, both results will be produced at the same time in either arrangement.

$$\begin{array}{ll}
25b^2 - 60abm^2 + 36a^2m^4 & (\pm 5b \mp 6am^2 \\
25b^2 & \\
\hline
* \quad -60abm^2 + 36a^2m^4 & (\pm 10b \mp 6am^2 \\
\quad\;\; -60abm^2 + 36a^2m^4 & \\
\hline
\qquad * \qquad\quad * &
\end{array}$$

In dividing $-60abm^2$ by $\pm 10b$, both signs are changed, the + to —, and the — to +. This gives to the second term the sign $\mp$. The first value is $5b - 6am^2$, and the second is $6am^2 - 5b$.

When the quantity whose second root is to be found, consists of more than three terms, it is not the second power of a binomial, but of a quantity consisting of more than two terms. Suppose the root to consist of the three terms $m + n + p$. If we represent the two first terms $m + n$ by l, the expression becomes $l + p$, the second power of which is

$$l^2 + 2lp + p^2.$$

Developing the second power l^2 of the binomial $m + n$, it becomes $m^2 + 2mn + n^2$. This shows that when the quantity is arranged according to the powers of some letter, the second root of the first term will be the first term m of the root. If m^2 be subtracted, and the next term be divided by $2m$, the next term n of the root will be obtained. If the second power of $m + n$ or l^2 be subtracted, the remainder will be $2lp + p^2$. If the next term $2lp$ be divided by $2l$ equal to twice $m + n$, the quotient will be p, the third term of the root. The same principle will extend to any number of terms.

It is required to find the second root of

$$4a^4 + 12a^3x + 13a^2x^2 + 6ax^3 + x^4.$$

Let this be disposed according to the powers of a or of x.

$x^4 + 6ax^3 + 13a^2x^2 + 12a^3x + 4a^4$ ($x^2 + 3ax + 2a^2$ root.
x^4

—— 1st dividend.
* $6ax^3 + 13a^2x^2$ ($2x^2 + 3ax$ 1st divisor.
$6ax^3 + 9a^2x^2$

2d divid. * $4a^2x^2 + 12a^3x + 4a^4$ ($2x^2 + 6ax + 2a^2$ 2d. di.
$4a^2x^2 + 12a^3x + 4a^4$

* * *

The process is so similar to that of numeral quantities that it needs no farther explanation.

The double sign need not be given to the terms during the operation. All the signs may be changed when the work is done, if the other root is wanted. This will seldom be the case when all the terms are positive; but when some of the terms are negative, if it is not known which quantities are the largest, the negative root is as likely to be found first as the positive. When this happens the positive will be found by changing all the signs.

Examples.

1. What is the second root of

$$4a^3x + 6a^2x^2 + a^4 + x^4 + 4ax^3?$$

2. What is the second root of

$$\frac{3x^2}{2} - \frac{1x}{2} + \frac{1}{16} + x^4 - 2x^3?$$

3. What is the second root of

$$-4x^4 + 4x^6 + 12x^3 - 6x + x^2 + 9?$$

4. What is the second root of

$$x^6 + 20x^3 + 25x^2 + 16 + 4x^5 + 10x^4 + 24x?$$

XXXVIII. *Extraction of the Roots of Compound Quantities of any Degree.*

By examining the several powers of a binomial, and observing that the principle may be extended to roots consisting of more than two terms, we may derive a general rule for extracting roots of any degree whatever.

$$
\begin{array}{rl}
(a+x)^1 = & a+x \\
 & a+x \\
\hline
 & a^2+ax \\
 & \qquad\; ax+x^2 \\
\hline
(a+x)^2 = & a^2+2ax+x^2 \\
 & a+x \\
\hline
 & a^3+2a^2x+ax^2 \\
 & \qquad\quad a^2x+2ax^2+x^3 \\
\hline
(a+x)^3 = & a^3+3a^2x+3ax^2+x^3 \\
 & a+x \\
\hline
 & a^4+3a^3x+3a^2x^2+ax^3 \\
 & \qquad\quad a^3x+3a^2x^2+3ax^3+x^4 \\
\hline
(a+x)^4 = & a^4+4a^3x+6a^2x^2+4ax^3+x^4 \\
 & a+x \\
\hline
 & a^5+4a^4x+6a^3x^2+4a^2x^3+ax^4 \\
 & \qquad\quad a^4x+4a^3x^2+6a^2x^3+4ax^4+x^5 \\
\hline
(a+x)^5 = & a^5+5a^4x+10a^3x^2+10a^2x^3+5ax^4+x^5
\end{array}
$$

By examining these powers, we find that the first term is the first term of the binomial, raised to the power to which the binomial is raised. The second term consists of the first term of the binomial one degree lower than in the first term, multiplied by the number expressing the power of the binomial, and also by the second term of the binomial. This will hereafter be shown to be true in all cases.

The application will be most easily understood by a particular example.

Let it be required to extract the 5th root of the quantity

$32\,a^{10} - 80\,a^8\,b^3 + 80\,a^6\,b^6 - 40\,a^4\,b^9 + 10\,a^2\,b^{12} - b^{15}$ $(2\,a^2 - b^3$

$32\,a^{10}$

—— Dividend.

* $-\,80\,a^8\,b^3$ $80\,a^8$ divisor.

The quantity being arranged according to the powers of a, I seek the fifth root of the first term $32\,a^{10}$. It is $2\,a^2$. This I write in the place of the quotient in division. I subtract the fifth power of $2\,a^2$, which is $32\,a^{10}$, from the whole quantity. The remainder is

$$-\,80\,a^8\,b^3 + 80\,a^6\,b^6 - \&c.$$

The second term of the fifth power of the binomial $a + x$ being $5\,a^4\,x$ shows that if the second term in this case be divided by five times the 4th power of $2\,a^2$, the quotient will be the next term of the root. The 4th power of $2\,a^2$ is $16\,a^8$ and 5 times this $80\,a^8$. Now $-\,80\,a^8\,b^3$ being divided by $80\,a^8$ gives $-\,b^3$ for the next term of the root. Raising $2\,a^2 - b^3$ to the fifth power, it produces the quantity given. If the root contained more than two terms it would be necessary to subtract the 5th power of $2\,a^2 - b^3$ from the whole quantity; and then to find the next term of the root, divide the first term of the remainder by five times the 4th power of $2\,a^2 - b^3$. The first term only however would be used which would be the same divisor that was used the first time.

When the number expressing the root has divisors, the roots may be found more easily than to extract them directly. The second root of a^4 is a^2, the second root of which is a. Hence the 4th root may be found by two extractions of the second root. The second root of a^6 is a^3, or the 3d root of a^6 is a^2. Hence the 6th root may be found by extracting the 2d and 3d roots. The 8th root is found by three extractions of the 2d root, &c.

Examples.

1. What is the 3d root of

$$6x^5 + x^6 - 40x^3 + 96x - 64?$$

2. What is the third root of

$$15x^4 - 6x + x^6 - 6x^5 - 20x^3 + 15x^2 + 1?$$

3. What is the 4th root of

$$216a^2x^2 - 216ax^3 + 81x^4 + 16a^4 - 96a^3x?$$

4. What is the 5th root of

$$80x^3 - 40x^2 + 32x^5 - 80x^4 - 1 + 10x?$$

XXXIX. *Extraction of the Roots of Numeral Quantities of any Degree.*

By the above expression of the several powers, we may extract any root of a numeral quantity. Let us take a particular example.

What is the 5th root of 5,443,532,400,000?

In the first place we observe that the 5th power of 10 is 100000, and the 5th power of 100 is 10000000000. Therefore if the root contains a figure in the ten's place, it must be sought among the figures at the left of the first five places counting from the right. Also if the root contains a figure in the hundred's place, it must be sought at the left of the first ten figures. This shows that the number may be divided into periods of five figures each, beginning at the right. The number so prepared will stand

544,35324,00000 (340
243

Dividend. 3013 (405 Divisor.
544 35324

---

* 00000

In the first place I find the greatest 5th power in 544. It is 243, the root of which is 3. I write 3 in the root, and subtract 243, the 5th power of 3, from 244. The remainder must contain $5\,a^4\,x + 10\,a^3x^2 +$, &c. The 3, that part of the root already found, and which, by the number of periods, must be 300, answers to a in the formula. $5\,a^4$, that is, five times the fourth power of 300 will form only an approximate divisor, since the remainder consists of several terms besides $5\,a^4\,x$; still it will enable us to judge very nearly, and we shall find the right number after one or two trials. As the fourth power of 30 will have no significant figure below 10000, (we may consider 3 to be in the ten's place, with regard to the next figure to be found,) we may bring down only one figure of the next period to the remainder for the dividend, and use 5 times the fourth power of 3 for the divisor. The dividend is 3013 and the divisor 405. The dividend contains the divisor at least 6 times, but probably 6 is too large for the root. Try 5. This gives for the first two figures 35. Raise 35 to the 5th power and see if it is equal to 544,25324. It will exceed it. Therefore try 4. The fifth power of 34 is 544,35324. Hence 34 is right. Subtract this from the number, there is no remainder. There is still another period, but it contains no significant figure, therefore the next figure is 0, and the root is 340. The 5th power of 340 is 5,443,532,400,000. If there had been a remainder after subtracting the 5th power of 34, it would have been necessary to bring down the next figure of the number to it to form a dividend, and then to divide it by 5 times the 4th power of 34; and to proceed in all respects as before.

The process of extracting roots above the second is very tedious. A method of doing it by logarithms will hereafter be shown, by which it may be much more expeditiously performed.

Examples.

1. What is the 5th root of 15937022465957?

2. What is the 4th root of 36469158961?

For this, the fourth root may be extracted directly, or it may be done by two extractions of the second root. Let the learner do it both ways.

3. What is the 6th root of 481890304?

This may be done by extracting the 6th root directly, or by extracting first the second and then the third root. Let it be done both ways.

4. What is the 7th root of 13492928512?

XL. *Fractional Exponents and Irrational Quantities.*

The method explained above, Art. XXXVI, for extracting the roots of literal quantities, gives rise to fractional exponents, when they cannot be exactly divided by the number expressing the root. Since quantities of this kind frequently occur, mathematicians have invented methods of performing the different operations upon them in the same manner as if the roots could be found exactly; and thus putting off the actual extracting of the root until the last, if it happens to be most convenient. The expressions also may often be reduced to others much more simple, and whose roots may be more easily found.

It has been already observed that the root of a quantity consisting of several factors, is the same as the product of the roots of the several factors.

Hence $(a^6 b^3)^{\frac{1}{3}} = (a^6)^{\frac{1}{3}}.\ (b^3)^{\frac{1}{3}} = a^2 b.$

$$(a^3)^{\frac{1}{2}} = (a^2)^{\frac{1}{2}}.\ (a)^{\frac{1}{2}} = (a)^{\frac{1}{2}}.\ (a)^{\frac{1}{2}}.\ (a)^{\frac{1}{2}}$$

$$= a^{\frac{1}{2}}.\ a^{\frac{1}{2}}.\ a^{\frac{1}{2}} = a.\ a^{\frac{1}{2}} = a^{1\frac{1}{2}} = a^{\frac{3}{2}}.$$

We see that the same expression may be written in a great many different forms. The most remarkable of the above are

$$a^{\frac{3}{2}} = a^{1\frac{1}{2}} = a^{1 + \frac{1}{2}} = a.\ a^{\frac{1}{2}}$$

On this principle we may actually take the root of a part of the factors of a quantity when they have roots, and leave the roots of the others to be taken by approximation at a convenient time.

The quantity $(72\ a^3\ b^5\ c)^{\frac{1}{2}}$ may be resolved into factors thus.

$$(2 \times 36\ a^2\ a\ b^4\ b\ c)^{\frac{1}{2}} = (36\ a^2\ b^4)^{\frac{1}{2}}.\ (2\ a\ b\ c)^{\frac{1}{2}}.$$

The root of the first factor $36\, a^2\, b^4$ can be found exactly, and the expression becomes

$$6\, a\, b^2\, (2\, a\, b\, c)^{\frac{1}{2}}.$$

This expression is much more simple than the other, for now it is necessary to find the root of only $2\, a\, b\, c$.

The expression might have been put in this form,

$$(72)^{\frac{1}{2}}\, a^{\frac{3}{2}}\, b^{\frac{5}{2}}\, c^{\frac{1}{2}} = (36.2)^{\frac{1}{2}}\, a^{1\frac{1}{2}}\, b^{2\frac{1}{2}}\, c^{\frac{1}{2}} = 6.2^{\frac{1}{2}}\, a\, a^{\frac{1}{2}}\, b^2\, b^{\frac{1}{2}}\, c^{\frac{1}{2}}$$
$$= 6\, a\, b^2\, (2\, a\, b\, c)^{\frac{1}{2}}.$$

Examples.

1. Reduce $(16\, a^5\, b^4)^{\frac{1}{3}}$ to its simplest form.

Ans. $2\, a\, b\, (2\, a^2\, b)^{\frac{1}{3}}$.

2. Reduce $(54\, a\, x^7)^{\frac{1}{3}}$ to its simplest form.

3. Reduce $\left(\frac{18\, a\, m^2}{147\, b^3\, c}\right)^{\frac{1}{2}}$ to its simplest form.

Ans. $\frac{3\, m}{7\, b} \left(\frac{2\, a}{3\, b\, c}\right)^{\frac{1}{2}}$.

4. Reduce $(16\, a^3\, b^5 + 32\, a^2\, b^3\, m)^{\frac{1}{2}}$ to its simplest form.

$$(16\, a^3\, b^5 + 32\, a^2\, b^3\, m)^{\frac{1}{2}} = (16\, a^2\, b^2)^{\frac{1}{2}}\ \ (a\, b^2 + 2\, b\, m)^{\frac{1}{2}}$$

Ans. $4\, a\, b\, (a\, b^3 + 2\, b\, m)^{\frac{1}{2}}$

5. Reduce $\left(\frac{135\, a^4\, x^5 - 108\, a^7\, x^6\, c^3}{64\, m^3\, n^2}\right)^{\frac{1}{3}}$ to its simplest form.

Sometimes it is convenient to multiply a root by another quantity, or one root by another.

If it is required to multiply $(3\, a^2\, b)^{\frac{1}{2}}$ by $a\, b$, it may be expressed thus: $a\, b\, (3\, a^2\, b)^{\frac{1}{2}}$. But if it is required actually to unite them, $a\, b$ must first be raised to the second power, and the product becomes $(3a^4b^3)^{\frac{1}{2}}$. This will appear more plain in the following manner,

$$(3a^2b)^{\frac{1}{2}} = 3^{\frac{1}{2}}\ a\ b^{\frac{1}{2}}.$$

This multiplied by $a\,b$ is

$$3^{\frac{1}{2}}\ a\,b^{\frac{1}{2}} \times a\,b = 3^{\frac{1}{2}}\ a^2\ b^{1\frac{1}{2}} = 3^{\frac{1}{2}}\ a^2\ b^{\frac{3}{2}} = (3a^4b^3)^{\frac{1}{2}}.$$

If instead of enclosing the quantity in the parenthesis and writing the exponent of the root over it, we divide the exponent of all the factors by the exponent of the root, all the operations will be very simple.

Let $a^{\frac{1}{2}}$ be multiplied by $a^{\frac{1}{2}}$.

$$a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a.$$

$$a^{\frac{1}{2}} \times a^{\frac{3}{2}} = a^{\frac{1}{2}+\frac{3}{2}} = a^{\frac{4}{2}} = a^2.$$

$$a^{\frac{2}{3}}\ b^{\frac{3}{5}} \times a^{\frac{5}{3}}\ b^{\frac{4}{5}} = a^{\frac{2}{3}+\frac{5}{3}}\ b^{\frac{3}{5}+\frac{4}{5}} = a^{\frac{7}{3}}\ b^{\frac{7}{5}}.$$

That is, multiplication is performed on similar quantities by adding the exponents, as when the exponents are whole numbers. In like manner division is performed by subtracting the exponents.

$$\frac{a^{\frac{7}{3}}}{a^{\frac{5}{3}}} = a^{\frac{7}{3}-\frac{5}{3}} = a^{\frac{2}{3}}.$$

It must be observed that $a^{\frac{2}{3}}$ may be read, *the third root of the second power of a*, or *the second power of the third root of a.* For the 3d root of a^2 is $a^{\frac{2}{3}}$; and

$$a^{\frac{1}{3}} \times a^{\frac{1}{3}} = a^{\frac{1}{3}+\frac{1}{3}} = a^{\frac{2}{3}}.$$

The 3d power of $a^{\frac{2}{5}}$ is

$$a^{\frac{2}{5}} \times a^{\frac{2}{5}} \times a^{\frac{2}{5}} = a^{\frac{2}{5}+\frac{2}{5}+\frac{2}{5}} = a^{\frac{2}{5}\times 3} = a^{\frac{6}{5}}.$$

That is, a power of a root may be found by multiplying the fractional exponent by the exponent of the power.

Consequently a root of a root may be found by dividing the fractional index by the exponent of the root. In multiplying and dividing the fractional exponents, we must apply the same rules that we apply to common fractions.

The 3d root of $a^{\frac{3}{5}}$ is $a^{\frac{1}{5}}$.

The 3d root of $a^{\frac{5}{2}}$ is $a^{\frac{5}{6}}$.

The 5th root of $a^{\frac{2}{3}}\ b^{\frac{3}{4}}$ is $a^{\frac{2}{15}}\ b^{\frac{3}{20}}$.

If the numerator and denominator both be multiplied or divided by the same number, the value of the quantity will not be altered; for that is the same as raising it to a power, and then extracting the root.

$$a^{\frac{2}{3}} = a^{\frac{4}{6}} = a^{\frac{10}{15}}.$$

If it is required to multiply $a^{\frac{2}{3}}$ by $a^{\frac{1}{2}}$, the fractions may be reduced to a common denominator and added: thus,

$$a^{\frac{2}{3}} \times a^{\frac{1}{2}} = a^{\frac{4}{6}} \times a^{\frac{3}{6}} = a^{\frac{7}{6}} = a^{1\frac{1}{6}} = a\ \ a^{\frac{1}{6}}.$$

The same may be done in division and the exponents subtracted.

$$\frac{a^{\frac{7}{2}}}{a^{\frac{5}{3}}} = \frac{a^{\frac{21}{6}}}{a^{\frac{10}{6}}} = a^{\frac{21}{6} - \frac{10}{6}} = a^{\frac{11}{6}}.$$

$$\frac{a^{\frac{3}{4}}}{a^{\frac{5}{3}}} = \frac{a^{\frac{9}{12}}}{a^{\frac{20}{12}}} = a^{\frac{9}{12} - \frac{20}{12}} = a^{-\frac{11}{12}} = \frac{1}{a^{\frac{11}{12}}}.$$

In fact, quantities with fractional exponents are subject to precisely the same rules, as when the exponents are whole numbers; but the rules must be applied as to fractions. The fractions may be reduced to decimals without altering the value; thus

$$a^{\frac{5}{4}} = a^{1\frac{1}{4}} = a^{1.25} = a \times a^{.25} = a \times a^{.2} \times a^{.05}$$

$$= a \times a^{\frac{2}{10}} \times a^{\frac{5}{100}}.$$

$$a^{\frac{7}{4}} \times a^{\frac{3}{5}} = a^{1.75} \times a^{.6} = a^{2.35} = a^{2} \times a^{\frac{3}{10}} \times a^{\frac{5}{100}}.$$

It is very important to remember how these quantities may be separated into factors. Since multiplication is performed by adding the exponents, and division by subtracting them, any quantity may be separated into as many factors as we please, by separating the exponent into parts. Thus,

$$a^{5} = a^{3} \times a^{2} = a \times a^{4} = a \times a^{2} \times a^{2}$$

$$= a^{\frac{2}{3}} \times a^{\frac{1}{3}} \times a^{2} \times a^{\frac{1}{2}} \times a^{\frac{1}{2}} \times a^{\frac{2}{5}} \times a^{\frac{3}{5}}.$$

The sum of all the exponents in the last expression is 5. Logarithms are of the same nature as these exponents, and afford as great a facility in operating upon numbers, as these do upon letters. And the operations are performed in the same way, as will be explained hereafter.

If the learner should ever have occasion to read other treatises on mathematics, he will generally find the roots expressed by what are called *radical signs.* The second root is expressed with the sign $\sqrt{}$, the third root $\sqrt[3]{}$ the same sign with the index of the root over it. The 4th root is $\sqrt[4]{}$, &c.

$$a^{\frac{1}{2}} = \sqrt{a}$$

$$a^{\frac{1}{3}} = \sqrt[3]{a}$$

$$a^{\frac{1}{4}} = \sqrt[4]{a}$$

$$a^{\frac{3}{5}} = \sqrt[5]{a^{3}}$$

$$2^{\frac{1}{3}}\ a^{\frac{2}{3}}\ b^{\frac{5}{3}} = \sqrt[3]{2\,a^{2}\,b^{5}}, \text{ \&c.}$$

They will be easily understood if the radical sign be removed, and the exponents divided by the index of the root or the quantity enclosed in a parenthesis, and the root written over it. The expression $\sqrt[4]{5\,a^{2}\,b^{3}}$ becomes

$$5^{\frac{1}{4}}\ a^{\frac{2}{4}}\ b^{\frac{3}{4}} = (5\,a^{2}\,b^{3})^{\frac{1}{4}}.$$

The expression $\sqrt{a^{2} + b^{2}}$ is equivalent to $(a^{2} + b^{2})^{\frac{1}{2}}$

XLI. *Binomial Theorem.*

It has already been remarked that the powers of any quantity are found by multiplying the quantity into itself as many times, less one, as is expressed by the exponent of the power. Sir Isaac Newton discovered a method, by which any quantity consisting of more than one term may be raised to any power whatever, without going through the process of multiplication.

The principle on which this method is founded is called the *Binomial Theorem.* Its use is very important and extensive in algebraic operations.

Next to quantities consisting of only one term, binomials, or quantities consisting of two terms, are the most simple.

Let a few of the powers of $a + x$ be found and their formation attended to.

$$\begin{array}{rl}
(a+x)^1 = & a + x \\
 & a + x \\
\cline{2-2}
 & a^2 + ax \\
 & \qquad ax + x^2 \\
\cline{2-2}
(a+x)^2 = & a^2 + 2ax + x^2 \\
 & a + x \\
\cline{2-2}
 & a^3 + 2a^2x + ax^2 \\
 & \qquad a^2x + 2ax^2 + x^3 \\
\cline{2-2}
(a+x)^3 = & a^3 + 3a^2x + 3ax^2 + x^3 \\
 & a + x \\
\cline{2-2}
 & a^4 + 3a^3x + 3a^2x^2 + ax^3 \\
 & \qquad a^3x + 3a^2x^2 + 3ax^3 + x^4 \\
\cline{2-2}
(a+x)^4 = & a^4 + 4a^3x + 6a^2x^2 + 4ax^3 + x^4 \\
 & a + x \\
\cline{2-2}
 & a^5 + 4a^4x + 6a^3x^2 + 4a^2x^3 + ax^4 \\
 & \qquad a^4x + 4a^3x^2 + 6a^2x^3 + 4ax^4 + x^5 \\
\cline{2-2}
(a+x)^5 = & a^5 + 5a^4x + 10a^3x^2 + 10a^2x^3 + 5ax^4 + x^5.
\end{array}$$

The law of the formation of the literal part is sufficiently manifest.

In each power there is one term more than the number denoting the power to which it is raised. The first power consists of two terms, the second power of three terms, the third power of four terms, &c.

In every power a is found in every term except the last, and x is found in every term except the first. The exponent of a in the first term is the same as the exponent of the power to which the binomial is raised, and it diminishes by one in each succeeding term.

The exponent of x in the second term is 1, and it increases by one in each succeeding term, until in the last term it is the same as that of a in the first term.

The law of the coefficients is not so simple, though it is not less remarkable.

The coefficients of the first power, viz. $a + x$, are 1, 1; those of the second power are 1, 2, 1. These are formed from the first as follows. When a is multiplied by a, it produces a^2, and no other term being produced like it, there is nothing added to it, and it remains with the same coefficient as the a in the multiplicand. In multiplying x by a and afterward a by x, two similar terms are produced, having the coefficients of the a and x in the multiplicand, viz. 1 and 1; and the addition of these forms the 2. The other 1 is produced like the first.

The coefficients of the third power are 1, 3, 3, 1. The 1s are produced from the second power, as those of the second power are produced from the first. In multiplying $2\,a\,x$ by a, the term produced is $2\,a^2\,x$, having the coefficient of the second term of the multiplicand; and in multiplying a^2 by x, the term produced is $a^2\,x$, similar to the last, and having the coefficient 1 of the first term of the multiplicand. The addition of the coefficients of these two terms produces the 3 before $a^2\,x$. That is, the coefficient of the second term of the third power is formed by adding together the coefficients of the first and second terms of the second power. In the same manner it may be shown, that the coefficient 3 of the third term of the third power is formed by adding together the coefficients of the second and third terms of the second power.

The following law will be found on examination to be general.

The coefficient of the first term of every power is 1. The coefficient of the second term of every power is formed by adding together the coefficients of the first and second terms of the preceding power. The coefficient of the third term of every power is formed by adding together the coefficients of the second and third terms of the preceding power. The coefficient of the fourth term of every power is found by adding together the coefficients of the third and fourth terms of the preceding power. And so of the rest.

This law, though perhaps sufficiently evident by inspection, may be easily demonstrated.

Suppose the above law to hold true as far as some power which we may designate by n. The literal part of the nth power will be formed thus.

$$a^n,\ a^{n-1}x,\ a^{n-2}x^2,\ a^{n-3}x^3 \ldots\ldots\ldots\ a\,x^{n-1},\ x^n.$$

We cannot write all the terms without assigning a particular value to n. We can write a few of the first and last. The points between show that the number of terms is indeterminate; there may or may not be more than are written.

Suppose that A is the coefficient of the second term, B that of the third, &c. and let the whole be multiplied by $a + x$, which will produce the next higher power, or the $(n + 1)$th power.

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$$\begin{array}{l}
a^n + A\,a^{n-1}x + B\,a^{n-2}x^2 + C\,a^{n-3}x^3 + \ldots\ldots F\,a\,x^{n-1} + x^n \\
a + x \\
\hline
a^{n+1} + A\,a^n x + B\,a^{n-1}x^2 + C\,a^{n-2}x^3 + \ldots\ldots F\,a^2x^{n-1} + a\,x^n \\
\qquad\qquad a^n x + A\,a^{n-1}x^2 + B\,a^{n-2}x^3 + C\,a^{n-3}x^4 \ldots\ldots F\,a\,x^n + x^{n+1} \\
\hline\hline
a^{n+1} + (1+A)a^n x + (A+B)\,a^{n-1}x^2 + (B+C)\,a^{n-2}x^3 + (C+)\,a^{n-3}x^4 \ldots\ldots (+F)\,a^2x^{n-1} + (F+1)ax^n + x^{n+1}.
\end{array}$$

In this result we observe that the exponents of both a and x are increased by 1 in each term, and there is still one term without x and another without a. Before the terms of the product were added, there were twice as many terms in the product as in the multiplicand, but they have all united two by two except the first and last. The terms $\mathrm{C}\,a^{n-3}\,x^4$ and $\mathrm{F}\,a^2\,x^{n-1}$ have not united with any others, but it is evident that they would have done so, if all the terms could have been written. There is then one more term in this power than in the last.

The coefficient of the first term is still 1. That of the second is the sum of the coefficients of the first and second terms of the multiplicand, viz. $1 + \mathrm{A}$. That of the third is the sum of the coefficients of the second and third terms of the multiplicand, viz. $\mathrm{A} + \mathrm{B}$; &c.

The above formula shows that if the law above mentioned is true for one power, it will be so for the next higher power. We have seen that it is true for the 5th power, therefore it will be true for the 6th; being true for the 6th, it will be so for the 7th, &c.

Let the coefficients of several of the first powers be written without the letters, forming them by the above principle.

First observe that $(a + x)^0 = 1$.

Adding 0 to this 1 gives 1, and then 0 again on the other side gives 1. Hence we have 1, 1 for the coefficients of the first power.

Adding 0 to the first 1 gives 1; adding 1 and 1 gives 2, and then 1 and 0 are 1. Hence the coefficients of the second power are 1, 2, 1.

Again, $0 + 1 = 1$; $1 + 2 = 3$; $2 + 1 = 3$; $1 + 0 = 1$. Hence 1, 3, 3, 1 are the coefficients of the third power.

Again, $0 + 1 = 1$; $1 + 3 = 4$; $3 + 3 = 6$; $3 + 1 = 4$; and $1 + 0 = 1$. Hence 1, 4, 6, 4, 1 are the coefficients of the fourth power.

Again, $0 + 1 = 1$; $1 + 4 = 5$; $4 + 6 = 10$; $6 + 4 = 10$; $4 + 1 = 5$; and $1 + 0 = 1$. Hence 1, 5, 10, 10, 5, 1 are the coefficients of the 5th power, &c.

The Coefficients of the first Ten Powers.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1

Here we observe that the first row of figures taken obliquely downward is the series of numbers 1, 1, 1, &c.

The second row is the series of natural numbers, 1, 2, 3, 4, 5, &c. whose differences are 1.

The third row is the series 1, 3, 6, 10, 15, &c. whose differences are the last series, viz. 1, 2, 3, 4, &c.

The fourth row is the series 1, 4, 10, 20, 35, &c. whose differences are the last series, viz. 1, 3, 6, 10, &c. Each successive row is a series, whose differences form the preceding row.

We may observe farther that the coefficient of the second term of any power is the term of the series 1, 2, 3, 4, &c. denoted by the exponent of the power. That of the second power, is the second term; that of the third power, the third term; that of the nth power, the nth term. But this being the series of natural numbers, the number which denotes the place of the term is equal to the term itself, so that the coefficient of the second term will always be equal to the exponent of the power.

The coefficient of the third term of any power is the term of the series 1, 3, 6, 10, &c. denoted by the exponent of the power diminished by 1. That of the third power is the second term, that of the fourth power the third term, that of the nth power the $(n-1)$th term, &c.

The coefficient of the fourth term of any power is the term of the series 1, 4, 10, 20, &c. denoted by the exponent of the power diminished by 2. That of the fourth power is the second term, that of the fifth power is the third term, that of the nth power is the $(n-2)$th term. And so on as we proceed to the right, the place of the term in the series is diminished by 1.

We may observe another remarkable fact, the reason of which will be manifest on recurring to the formation of these series. We shall take the 7th power for an example, though it is equally true of any other.

The coefficient of the second term, viz. 7, is the sum of 7 terms of the preceding series 1, 1, 1, &c. and was in fact formed by adding them.

The coefficient of the third term, 21, is the sum of the first six terms of the preceding series, 1, 2, 3, &c. and was actually formed by adding them, as may be seen by referring to the formation.

The coefficient of the fourth term, 35, is the sum of the first five terms of the preceding series, 1, 3, 6, 10, &c. and was formed by adding them.

The same law continues through the whole. If now we can discover a simple method of finding the sums of these series without actually forming the series themselves, it will be easy to find the coefficients of any power without forming the preceding powers. This will be our next inquiry.

XLII. *Summation of Series by Differences.*

It is not my purpose at present to enter very minutely into the theory of series. I shall examine only a few of the most simple of them, and those principally with a view of demonstrating the *binomial theorem.*

A series by differences is several numbers arranged together, the successive terms of which differ from each other by some regular law.

I call a series of the first order that, in which all the terms are alike, as 1, 1, 1, 1, &c. 3, 3, 3, 3, &c. a, a, a, a, &c. In these the difference is zero.

The sum of all the terms of such a series is evidently found by multiplying one of the terms by the number of terms in the series. Every case of multiplication is an example of finding the sum of such a series.

The sum s of a number n of terms of any series a, a, a, &c. is expressed

$$s = \frac{na}{1}.$$

When $a = 1$, it becomes $s = \frac{n}{1}$.

A series in which the terms increase or diminish by a constant difference, is called a series of the *second order*. As 1, 2, 3, 4, 5, &c. 3, 6, 9, 12, &c. or 12, 9, 6, 3. A series of this kind is formed from a series of the first order. The differences between the successive terms form the series from which it is derived.

At present I shall examine only the series of natural numbers 1, 2, 3, 4, ... n.

This series is formed as follows:

$$\begin{aligned} 0+1&=1\\ 1+1&=2\\ 1+1+1&=3\\ 1+1+1+1&=4\\ 1+1+1+1+1&=5, \text{\&c.} \end{aligned}$$

The sum of any number n of terms of the series 1, 1, 1, 1, &c. is equal to the nth term of the series 1, 2, 3, 4, &c.

Write down two of these series as follows and add the corresponding terms of the two together.

$$\begin{array}{ccccc} 1, & 2, & 3, & 4, & 5\\ 5, & 4, & 3, & 2, & 1\\ \hline 6, & 6, & 6, & 6, & 6 \end{array}$$

$$\begin{array}{cccccccc}
1, & 2, & 3, & 4, \ldots & (n-3), & (n-2), & (n-1,) & n \\
n, & (n-1), & (n-2), & (n-3) \ldots & 4, & 3, & 2, & 1 \\
\hline
(n+1), & (n+1), & (n+1), & (n+1) \ldots & (n+1), & (n+1), & (n+1), & (n+1)
\end{array}$$

The 6th term of the series is 6, and it appears that 5 times 6 will be twice the sum of 5 terms of the series.

The $(n+1)$th term of the series 1, 2, 3, 4, &c. is $n+1$. It appears that n times $(n+1)$ will be twice the sum of n terms of the series.

The sum s' of any number n of terms may be expressed thus.

$$s' = \frac{n(n+1)}{1.2}.$$

It is frequently convenient to use the same letter in similar situations to express different values. In order to distinguish it in different places, it may be marked thus, s, s', s'', s''', which may be read s, s prime, s second, s third, &c.

How many times does the hammer of a clock strike in 12 hours?

In this example $n = 12 \quad n + 1 = 13$.

$$\frac{12 \times 13}{1 \times 2} = 78. \qquad \textit{Ans. } 78 \text{ times.}$$

The rule expressed in words is; *To find the sum of any number of terms of the series* 1, 2, 3, 4, &c. *find the next succeeding term in the series, and multiply it by the number of terms in the series, and divide the product by* 2.

The same thing may be proved in another form which is more conformable to the method that will be used for the series of the higher orders.

Suppose it is required to find the sum of the first five terms of the series.

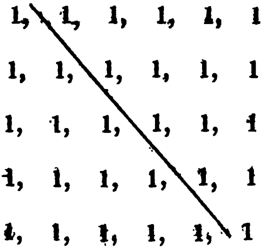
The sixth term of the series is the sum of 6 terms of the series, 1, 1, 1, &c. thus

$$1+1+1+1+1+1 = 6.$$

Let this series be written down five times, one under the other, thus.

1, 1, 1, 1, 1, 1
1, 1, 1, 1, 1, 1
1, 1, 1, 1, 1, 1
1, 1, 1, 1, 1, 1
1, 1, 1, 1, 1, 1

If this series be divided by a line passing diagonally through it, so that the part below and at the left of the line may contain one term of the first series, two of the second, three of the third, four of the fourth, and five of the fifth; the terms so separated will form the first five terms of the series 1, 2, 3, &c. There will be the same number of terms above and at the right of the line, which will form the same series, if the terms be added vertically instead of horizontally.



It is easy to see that this series continued to any number of terms will be formed twice over in this way, if the number of series written under each other is equal to the number of terms required and the number of terms in each series exceed the number of terms by one. And the reason of it is manifest from the manner in which the two series are formed.

Hence n times the series consisting of $n + 1$ terms of the series 1, 1, 1, 1, &c. will be twice the sum s' of n terms of the series 1, 2, 3, 4, &c.

That is, $2\,s' = n\,(n + 1)$ and $s' = \frac{n\,(n+1)}{2}$.

A series of *the third order* is one, the difference of the successive terms of which is a series of the second order. I shall consider only the series formed from the series 1, 2, 3, &c.

Formation.

$$\begin{aligned}
0+1 &= 0+1=1\\
1+2 &= 1+2=3\\
1+2+3 &= 3+3=6\\
1+2+3+4 &= 6+4=10\\
1+2+3+4+5 &= 10+5=15\\
1+2+3+4+5+6 &= 15+6=21, \text{\&c.}
\end{aligned}$$

The first term of the series 1, 2, 3, &c. forms the first term; the sum of the first two terms forms the second; the sum of the first three forms the third term, &c. and the sum of n terms will form the nth term of the series 1, 3, 6, 10, &c.

Let it be required to find the sum of the first five terms of the series 1, 3, 6, 10, 15, 21, &c.

The sixth term of this series is the sum of the first 6 terms of the series 1, 2, 3, &c.

$$1+2+3+4+5+6=\frac{6\times 7}{2}=21=6\text{th term.}$$

Write this series five times one under the other, and draw a line diagonally so as to leave on the left and below, the first term of the first, the first two of the second, the first three of the third, &c. and the first five of the fifth.

$$\begin{array}{cccccc}
1, & 2, & 3, & 4, & 5, & 6\\
1, & 2, & 3, & 4, & 5, & 6\\
1, & 2, & 3, & 4, & 5, & 6\\
1, & 2, & 3, & 4, & 5, & 6\\
1, & 2, & 3, & 4, & 5, & 6
\end{array}$$

The figures so cut off form the first five terms of the series 1, 3, 6, 10, 15, &c. the sum of which we wish to find. It will now be shown that the sum of the terms on the right and above the line, is equal to twice the sum of those below and at the left.

By the rule given above for finding the sum of the series 1, 2, 3, &c.

The sum of 1 term, or 1 $= \frac{1 \times 2}{2}$.

The sum of 2 terms, or $1 + 2$ $= \frac{2 \times 3}{2}$.

The sum of 3 terms, or $1 + 2 + 3$ $= \frac{3 \times 4}{2}$.

The sum of 4 terms, or $1 + 2 + 3 + 4$ $= \frac{4 \times 5}{2}$.

The sum of 5 terms, or $1 + 2 + 3 + 4 + 5$ $= \frac{5 \times 6}{2}$.

Hence $2\,(1) = 1 \times 2$
$2\,(1 + 2) = 2 \times 3$
$2\,(1 + 2 + 3) = 3 \times 4$
$2\,(1 + 2 + 3 + 4) = 4 \times 5$
$2\,(1 + 2 + 3 + 4 + 5) = 5 \times 6$

That is, the 2 is twice the 1,
The two threes are twice $(1 + 2)$,
The three fours are twice $(1 + 2 + 3)$,
The four fives are twice $(1 + 2 + 3 + 4)$, and
The five sixes are twice $(1 + 2 + 3 + 4 + 5)$.

Since the part below the line forms the series whose sum is required, and the part above the line is equal to twice that below, both parts together are equal to three times the series 1, 3, 6, 10, 15. Therefore if 21, which is the next term in the series, and which is also the sum of the series 1, 2, 3, 4, 5, 6, be multiplied by 5, the number of terms to be summed, and divided by 3, the quotient will be the sum of the series required.

It is easy to see that if the series 1, 2, 3, ... $(n+1)$ be written n times and divided by a line like the above, the part below the line will form n terms of the series 1, 3, 6, 10, &c. And the part above the line will be equal to twice the part below, because the sum of n terms of the series 1, 2, 3, &c. is

$$\frac{n(n+1)}{1\times 2}.$$

Therefore to find the sum of n terms of the series 1, 3, 6, 10, multiply the $(n+1)$th term of that series by n and divide by 3, and the quotient will be the sum required.

But the $(n+1)$th term of the series is equal to the sum of $(n+1)$ terms of the series 1, 2, 3, 4, &c. The nth term of this series being $\frac{n(n+1)}{1\times 2}$, the $(n+1)$th term will be

$$\frac{(n+1)(n+2)}{1\times 2}.$$

This being multiplied by n, the number of terms, and divided by 3, gives

$$\frac{n(n+1)(n+2)}{1\times 2\times 3}.$$

Hence the sum s'' of n terms of the series will be expressed thus,

$$s''=\frac{n(n+1)(n+2)}{1\times 2\times 3}.$$

A series of the fourth order is one, the difference of whose terms is a series of the third order.

I shall at present consider only the one formed from the series 1, 3, 6, 10, 15, &c.

Formation.

$0 + 1 = 0 + 1 = 1$
$1 + 3 = 1 + 3 = 4$
$1 + 3 + 6 = 4 + 6 = 10$
$1 + 3 + 6 + 10 = 10 + 10 = 20$
$1 + 3 + 6 + 10 + 15 = 20 + 15 = 35$
$1 + 3 + 6 + 10 + 15 + 21 = 35 + 21 = 56$

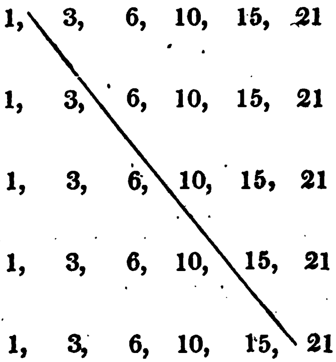
The first term of the series 1, 3, 6, &c. is the first term of the new series; the sum of the first two terms forms the second; &c. the sum of *n* terms will form the *n*th term of the new series.

It is required to find the sum of five terms of this series.

The sixth term of this series is equal to the sum of the first six terms of the preceding.

$$1 + 3 + 6 + 10 + 15 + 21 = \frac{6 \times 7 \times 8}{1 \times 2 \times 3} = 56.$$

Write this series five times, one under the other, and separate it into two parts by a line drawn diagonally in the same manner as was done with the last series. The terms below the line will form the series whose sum is required, and the terms above the line will be equal to three times those below. That is, the whole will be four times the sum required.



1, 3, 6, 10, 15, 21
1, 3, 6, 10, 15, 21
1, 3, 6, 10, 15, 21
1, 3, 6, 10, 15, 21
1, 3, 6, 10, 15, 21

By the rule given above for finding the sum of the series 1, 3, 6, 10, &c.

The sum of one term, or 1 $\frac{1\times3}{3}=1.$

The sum of two terms, or $1+3$ $=\frac{2\times6}{3}=4.$

The sum of three terms, or $1+3+6$ $=\frac{3\times10}{3}=10.$

The sum of four terms, or $1+3+6+10=\frac{4\times15}{3}=20.$

The sum of five terms, or $1+3+6+10+15=\frac{5\times21}{3}=35.$

The five 21s are 3 times $1+3+6+10+15.$

The four 15s are 3 times $1+3+6+10$

and so of the rest.

It is easy to see that this principle will extend to any number of terms.

Therefore to find the sum of n terms of the series 1, 4, 10, 20, &c., multiply the $(n+1)$th term of the series by n, and divide the product by 4, and the quotient will be the sum required.

But the $(n+1)$th term of this series is equal to the sum of $(n+1)$ terms of the preceding series.

The nth term of the preceding series being

$$\frac{n(n+1)(n+2)}{1\times2\times3},$$

the $(n+1)$th term will be

$$\frac{(n+1)(n+2)(n+3)}{1\times2\times3}.$$

This being multiplied by n and divided by 4, gives

$$s'''=\frac{n(n+1)(n+2)(n+3)}{1\times2\times3\times4}.$$

XLIII. The principle of summing these series may be proved generally as follows:

Let 1, a, b, c, d . . . . . . l be a series of any order, such that the sum of n terms may be found by multiplying the $(n+1)$th term by n, and dividing the product by m. If l is the $(n+1)$th term, and s the sum of all the terms, we shall have by hypothesis

$$s = \frac{n\,l}{m}, \text{ and } m\,s = n\,l.$$

That is, $n\,l$ will be m times the sum of the series. The next higher series will be formed from this as follows:

1 . . . . . . $=$ 1st term.
$1 + a$ . . . . . $=$ 2d "
$1 + a + b$ . . . . $=$ 3d "
$1 + a + b + c$ . . . $=$ 4th "
$1 + a + b + c + d$ . . $=$ 5th "

. . . .

$1 + a + b + c + d + \ldots . k$ $=$ nth "
$1 + a + b + c + d + \ldots . k + l =$ $(n+1)$th.

The first term 1 of the original series 1, a, b, &c., forms the first term of the new series; the sum of the first two forms the second term; the sum of the first three forms the third term, &c., and the sum of $(n+1)$ terms forms the $(n+1)$th term.

Let the series forming the $(n+1)$th term, be written n times, one under the other, term for term. And let a line be drawn diagonally, so that the first term of the first row, the first two of the second row, and n terms of the nth row may be at the left, and below the line.

$$\begin{array}{cccccccc}
1, & \backslash\, a, & b, & c, & d, & . & k, & l \\
1, & a, & \backslash\, b, & c, & d, & . & k, & l \\
1, & a, & b, & \backslash\, c, & d, & . & k, & l \\
1, & a, & b, & c, & \backslash\, d, & . & k, & l \\
1, & a, & b, & c, & d, & \backslash\, . & k, & l \\
1, & a, & b, & c, & d, & . & \backslash\, k, & l \\
1, & a, & b, & c, & d, & . & k, & \backslash\, l
\end{array}$$

The terms below and at the left of the line, form n terms of the new series. It is now to be shown that the terms above, and at the right of the line, are equal to m times those below, and, consequently, that the whole together are equal to $m + 1$ times n terms of the new series.

By the hypothesis

The sum of one term, or $1 = \frac{1\,a}{m}$.

The sum of two terms, or $1 + a = \frac{2\,b}{m}$.

The sum of three terms, or $1 + a + b = \frac{3\,c}{m}$.

The sum of four terms, or $1 + a + b + c = \frac{4\,d}{m}$.

The sum of n terms, or $1 + a + b + c + d + \ldots k = \frac{n\,l}{m}$.

Multiplying both members of the above equations by m:

$$\begin{aligned}
m.\,1 &= 1\,a \\
m\,(1 + a) &= 2\,b \\
m\,(1 + a + b) &= 3\,c \\
m\,(1 + a + b + c) &= 4\,d \\
m\,(1 + a + b + c + d + \ldots k) &= n\,l
\end{aligned}$$

Hence it appears, that a is m times 1; $2\,b$ is m times $(1 + a)$ &c.; and $n\,l$ is m times $(1 + a + b + c + d + \ldots.\ k)$; that is, the part above and at the right of the line, is m times the part at the left and below; consequently the whole, or n times the $(n + 1)$th term of the new series, will be $(m + 1)$ times the sum of n terms of the same series.

We have already examined all the series as far as the fourth order, and have found the above hypothesis true so far. Let us suppose the series 1, a, b, &c. to be a series of the fourth order, in which we have found that the sum of n terms may be obtained by multiplying the $(n + 1)$th term by n, and dividing the product by 4; in this case m is equal to 4. The series formed from this will be a series of the 5th order, and $m + 1 = 4 + 1 = 5$. Therefore by the above demonstration it appears that the sum of n terms of a series of the 5th order may be obtained by multiplying the $(n + 1)$th term by n, and dividing the product by 5.

If now the series, 1, a, b, &c., be considered a series of the 5th order, $m = 5$ and $m + 1 = 6$. Hence the same principle extends to the 6th order.

If then we continue to make 1, a, b, &c., represent one series after another in this way, we shall see that the principle will extend to any order whatever of this kind of series.

We have then this general rule;

To find the sum of n terms of a series of the order denoted by r, derived from the series 1, 1, 1, &c., *multiply the $(n + 1)$th term of the series by n and divide the product by r.*

Also, the nth term of the series of the order r, is equal to the sum of n terms of the series of the order $r - 1$.

When the series is of the first order, the sum of n terms is $\frac{n.1}{1}$ or $\frac{n}{1}$

The sum of $(n + 1)$ terms of this series is $\frac{n+1}{1}$. This is the $(n + 1)$th term of the series of the second order. This multiplied by n and divided by 2 gives the sum of n terms of the series of the second order:

$$\frac{n\,(n+1)}{1 \times 2}.$$

The sum of $(n + 1)$ terms of the same series is

$$\frac{(n+1)\,(n+2)}{1 \times 2}$$

This is the $(n + 1)$th term of the series of the third order. This multiplied by n and divided by 3 gives the sum of n terms of this series:

$$\frac{n\,(n+1)\,(n+2)}{1 \times 2 \times 3}$$

The sum of $(n + 1)$ terms of the last series is

$$\frac{(n+1)\,(n+2)\,(n+3)}{1 \times 2 \times 3}$$

This is the $(n + 1)$th term of the series of the fourth order. This multiplied by n and divided by 4 gives the sum of n terms of the series of the fourth order:

$$\frac{n\,(n+1)\,(n+2)\,(n+3)}{1 \times 2 \times 3 \times 4}.$$

Hence for the series of the order r we have this formula:

$$\frac{n\,(n+1)\,(n+2)\,(n+3)\ldots\ldots(n+r-1)}{1 \times 2 \times 3 \times 4 \times \ldots\ldots r}.$$

We have examined only the series formed from the series 1, 1, 1, 1, &c., which are sufficient for our present purpose. The principle may be generalized so as to find the sum of any series of the kind, whatever be the original series, and whatever be the first terms of those formed from it.

XLIV. *Binomial Theorem.*

Before reading this article, it is recommended to the learner to review article XLI.

Let it now be required to find the 7th power of $a + x$. The letters without the coefficients stand thus;

$$a^7,\ a^6 x,\ a^5 x^2,\ a^4 x^3,\ a^3 x^4,\ a^2 x^5,\ a x^6,\ x^7.$$

The coefficient of the first term we observed Art. XLI, is always 1. That of the second term is 7, the exponent of the power, or the 7th term of the series 1, 2, 3, &c.

The coefficient of the third term is the sixth term of the series of the third order 1, 3, 6, 10, &c. which is the sum of six terms of the series 1, 2, 3, &c. This sum is found by multiplying the 7th term of the series by 6 and dividing the product by 2. But the 7th term is 7, the coefficient last found.

$$\frac{6 \times 7}{2} = 21.$$

The coefficient is 21.

The coefficient of the fourth term is the 5th term of the series 1, 4, 10, &c., or it is the sum of five terms of the preceding series. The sum of five terms of the series 1, 3, 6, &c., is found by multiplying the 6th term by 5 and dividing the product by 3. The 6th term is the coefficient last found, viz. 21.

$$\frac{5 \times 21}{3} = 35.$$

The coefficient is 35.

The coefficient of the fifth term is the fourth term of the series of the fifth order 1, 5, 15, &c., or it is the sum of 4 terms of the preceding series. The sum of 4 terms of the series 1, 4, 10, &c. is found by multiplying the fifth term of the series by 4 and dividing the product by 4. The fifth term is the coefficient last found, viz. 35.

$$\frac{4 \times 35}{4} = 35.$$

The coefficient is 35.

The coefficient of the 6th term is the 3d term of the series of the sixth order, which is the sum of 3 terms of the series of the 5th order. The sum of 3 terms of this series is found by multiplying the 4th term by 3 and dividing the product by 5. The 4th term is the coefficient last found, viz. 35

$$\frac{3 \times 35}{5} = 21.$$

The coefficient is 21.

The coefficient of the 7th term is the 2d term of the series of the 7th order, which is the sum of two terms of the series of the 6th order. The 3d term of this series is the coefficient last found, viz. 21.

$$\frac{2 \times 21}{6} = 7.$$

The coefficient is 7.

The coefficient of the last term is 1, though it may be found by the rule

$$\frac{1 \times 7}{7} = 1.$$

Hence the 7th power of $a + x$ is

$$a^7 + 7\,a^6 x + 21\,a^5 x^2 + 35\,a^4 x^3 + 35\,a^3 x^4 + 21\,a^2 x^5 + 7\,a x^6 + x^7$$

Examining the formation of the above coefficients, we observe, that each coefficient was found by multiplying the coefficient of the preceding term by the exponent of the leading quantity a in that term, and dividing the product by the number which marks the place of that term. Thus the coefficient of the third term was found by multiplying 7, the coefficient of the second term, by 6, the exponent of a in the second term, and dividing the product by 2, the number which marks the place of the second term. This will be true for all cases, because that exponent must necessarily show the number of terms of which the sum is to be found; the coefficient will always be the term to be multiplied, because the number of terms always diminishes by 1 for the successive coefficients, and the place of the term always marks the order of the series of which the sum is to be found.

Hence is obtained the following general rule.

Knowing the coefficient of any term in the power, the coefficient of the succeeding term is found by multiplying the coefficient of the known term by the exponent of the leading quantity in that term, and dividing the product by the number which marks the place of that term from the first.

The coefficient of the first term, being always 1, is always known. Therefore, beginning with this, all the others may be found by the rule.

It may be farther observed, that the coefficients of the last half of the terms, are the same as those of the first half in an inverted order. This is evident by looking at the coefficients, page 275, and observing that the series are the same, whether taken obliquely to the left or to the right.

It is also evident from this, that $a + x$ is the same as $x + a$, and that, taken from right to left, x is the leading quantity in the same manner as a is the leading quantity from left to right.

Hence it is sufficient to find coefficients of one half of the terms when the number of terms is even, and of one more than half when the number is odd. The same coefficients may then be written before the corresponding terms counted from the right.

In the above example of the 6th power, the coefficients of the first four terms being found, we may begin on the right and put 6 before the second, and 15 before the third, and then the power is complete.

Examples.

1. What is the 7th power of $a + x$?

Ans. $a^7 + 7a^6x + 21a^5x^2 + 35a^4x^3 + 35a^3x^4 + 21a^2x^5$
$+ 7ax^6 + x^7$.

2. What is the 10th power of $a + x$?

Ans. $a^{10} + 10a^9x + 45a^8x^2 + 120a^7x^3 + 210a^6x^4 + \ldots$
$252a^5x^5 + 210a^4x^6 + 120a^3x^7 + 45a^2x^8 + 10ax^9 + x^{10}$.

3. What is the 9th power of $a + b$?

4. What is the 13th power of $m + n$?

5. What is the 2d power of $2ac + d$?

Make $2ac = b$.

The 2d power of $b + d$ is $b^2 + 2bd + d^2$.

Putting $2ac$ the value of b into this, instead of b, observing that $b^2 = 4a^2c^2$, and it becomes

$$4a^2c^2 + 4acd + d^2.$$

6. What is the 3d power of $3c^2 + 2bd$?

Make $a = 3c^2$ and $x = 2bd$.

The 3d power of $a + x$ is $a^3 + 3a^2x + 3ax^2 + x^3$.

Put into this the values of a and x and it becomes

$$27c^6 + 54c^4bd + 36c^2b^2d^2 + 8b^3d^3,$$

which is the 3d power of $3c^2 + 2bd$.

7. What is the 3d power of $a - b$?

Make $x = -b$, then having found the 3d power of $a + x$ put $-b$ in the place of x and it becomes

$$a^3 - 3a^2b + 3ab^2 - b^3,$$

which is the 3d power of $a - b$.

In fact it is evident that the powers of $a - b$ will be the same as the powers of $a + b$, with the exception of the signs. It is also evident that every term which contains an odd power of the term affected with the sign — must have the sign —, and every term which contains an even power of the same quantity must have the sign +.

8. What is the 7th power of $m - n$?

9. What is the 4th power of $2a - bc^2$?

10. What is the 5th power of $a^3c - 2c^4$?

11. What is the 3d power of $a + b + c$?

Make $m = b + c$. Then $a + m = a + b + c$.

The 3d power of $a + m$ is $a^3 + 3\,a^2 m + 3\,a m^2 + m^3$.

But $m = b + c$, $m^2 = b^2 + 2\,b\,c + c^2$, and

$$m^3 = b^3 + 3\,b^2 c + 3\,b c^2 + c^3.$$

Substituting these values of m, the third power of $a + b + c$ will be

$$a^3 + 3a^2b + 3a^2c + 3\,a\,b^2 + 6\,a b c + 3\,a\,c^2 + b^3 + 3\,b^2\,c + 3bc^2 + c^3.$$

12. What is the 3d power of $a - b + c$?

Make $a - b = m$, raise $m + c$ to the 3d power, and then substitute the value of m.

Ans. $a^3 - 3\,a^2 b + 3\,a^2 c + 3\,a\,b^2 - 6\,a\,b\,c - 3\,a\,c^2 - b^3 \ldots\ldots + 3\,b^2\,c - 3\,b\,c^2 + c^3$;

which is the same as the last, except that the terms which contain the odd powers of b have the sign $-$.

Hence it is evident that the powers of any compound quantity whatever, may be found by the binomial theorem, if the quantity be first changed to a binomial with two simple terms, one letter being made equal to several, that binomial raised to the power required, and then the proper letters restored in their places.

13. What is the 2d power of $a + b + c - d$?

Ans. $a^2 + 2\,a\,b + b^2 + 2\,a\,c + 2\,b\,c - 2\,a\,d - 2\,b\,d + c^2 \ldots\ldots - 2\,c\,d + d^2$.

14. What is the 3d power of $2\,a - b + c^2$?

15. What is the 7th power of $3\,a^5 - 2\,a^2\,d$?

16. What is the 4th power of $7\,b^3 + 2\,c - d^3$?

17. What is the 13th power of $a^3 - 2\,b^2$?

18. What is the 5th power of $a^2 - a - 2\,d$?

19. What is the 3d power of $a - 2\,d + c^2\,d$?

20. What is the 3d power of $a - b - 2\,c^2 - d^3$?

21. What is the 5th power of $7\,a^2\,b^5 - 10\,a^6\,c^2$?

XLV. The rule for finding the coefficients of the powers of binomials may be derived and expressed more generally as follows:

It is required to find the coefficients of the nth power of $a + x$.

It has already been observed, Art. XLI., that the coefficient of the second term of the nth power is the nth term of the series of the second order, 1, 2, 3, &c., or, the sum of n terms of the series 1, 1, 1, &c.; that the coefficient of the third term is the sum of $(n - 1)$ terms of the series of the second order; that the coefficient of the fourth term is the sum of $(n - 2)$ terms of the series of the third order, &c. So that the coefficient of each term is the sum of a number of terms of the series of the order less by one, than is expressed by the place of the term; and the number of terms to be used is less by one for each succeeding series.

By Art. XLII. the sum of n terms of the series 1, 1, 1, is $\frac{n}{1}$ The sum of $(n - 1)$ terms of the series of the second order is

$$\frac{n(n-1)}{1 \times 2}.$$

The sum of $(n - 2)$ terms of the series of the third order is

$$\frac{n(n-1)(n-2)}{1 \times 2 \times 3}.$$

Hence $(a + x)^n = a^n + \frac{n}{1} a^{n-1} x + \frac{n(n-1)}{1 \times 2} a^{n-2} x^2$

$$+ \frac{n(n-1)(n-2)}{1 \times 2 \times 3} a^{n-3} x^3 + \&c.$$

It may be observed that n is the exponent of a in the first term, and that n or its equal $\frac{n}{1}$ forms the coefficient of the second term.

The coefficient of the third term is $\frac{n}{1}$ multiplied by $\frac{n-1}{2}$, or multiplied by $(n - 1)$ and divided by 2. But $(n - 1)$ is the exponent of a in the 2d term, and 2 marks the place of the second term from the left. Therefore the coefficient of the third term is found by multiplying the coefficient of the second term by the exponent of a in that term, and dividing the product by the number which marks the place of that term from the left.

By examining the other terms, the following general rule will be found true.

Multiply the coefficient of any term by the exponent of the leading quantity in that term, and divide the product by the number that marks the place of that term from the left, and you will obtain the coefficient of the next succeeding term. Then diminish the exponent of the leading quantity by 1 *and increase that of the other by* 1 *and the term is complete.*

By this rule only the requisite number of terms can be obtained. For x^n, which is properly the last term of $(a + x)^n$, is the same as $a^0\,x^n$. If we attempt by the rule to obtain another term from this, it becomes $0 \times a^{-1}\,x^{n+1}$ which is equal to zero.

It has been remarked above, that the coefficients of the last half of the terms of any power, are the same as those of the first reversed. This may be seen from the general expression:

If $n = 7$, then $\frac{n}{1} = \frac{7}{1}$; $\frac{n-1}{2} = \frac{6}{2}$; $\frac{n-2}{3} = \frac{5}{3}$;

$$\frac{n-3}{4} = \frac{4}{4} = 1\,; \quad \frac{n-4}{5} = \frac{3}{5}; \quad \frac{n-5}{6} = \frac{2}{6};$$

$$\frac{n-6}{7} = \frac{1}{7}.$$

This furnishes the following fractions, viz.

$$\tfrac{7}{1},\ \tfrac{6}{2},\ \tfrac{5}{3},\ \tfrac{4}{4},\ \tfrac{3}{5},\ \tfrac{2}{6},\ \tfrac{1}{7}.$$

The first of these is the coefficient of the second term; the coefficient of the second multiplied by $\frac{6}{2}$ forms the coefficient of the third term, &c.

$$\tfrac{7}{1} \times \tfrac{6}{2} = 21. \quad 21 \times \tfrac{5}{3} = 35.$$

Now 35 multiplied by $\frac{4}{4} = 1$ will not be altered; hence two successive coefficients will be alike. 21 multiplied by $\frac{5}{3}$ produced 35; so 35 multiplied by $\frac{3}{5}$ must reproduce 21. In this way all the terms will be reproduced; for the last half of the fractions are the first half inverted.

This demonstration might be made more general, but it is not necessary.

XLVI. *Progression by Difference, or Arithmetical Progression.*

A series of numbers increasing or decreasing by a constant difference, is called a *progression by difference*, and sometimes an *arithmetical progression*.

The first of the two following series is an example of an increasing, and the second of a decreasing, progression by difference.

$$5, \quad 8, \quad 11, \quad 14, \quad 17, \quad 20, \quad 23 \ldots\ldots.$$
$$50, \quad 45, \quad 40, \quad 35, \quad 30, \quad 25, \quad 20 \ldots\ldots.$$

It is easy to find any term in the series without calculating the intermediate terms, if we know the first term, the common difference, and the number of that term in the series reckoned from the first.

Let a be the first term, r the common difference, and n the number of terms. The series is

$$a, a + r, a + 2r, a + 3r \ldots\ldots a + (n - 2)\,r, a + (n - 1)\,r.$$

The points . . . . . are used to show that some terms are left out of the expression, as it is impossible to express the whole until a particular value is given to n.

Let l be the term required, then

$$l = a + (n - 1)\,r.$$

Hence, any term may be found by adding the product of the common difference by the number of terms less one, to the first term.

Example.

What is the 10th term of the series 3, 5, 7, 9, &c.

In this $a = 3$, $r = 2$, and $n - 1 = 9$.

$$l = 3 + 9 \times 2 = 21.$$

In a decreasing series, r is negative.

Example.

What is the 13th term of the series 48, 45, 42, &c.?

$$a = 48, r = -3, \text{ and } n - 1 = 12.$$

$$l = 48 + (12 \times -3) = 48 - 36 = 12.$$

Let a, b, c, be any three successive terms in a progression by difference.

By the definition,

$$b - a = c - b$$

$$2b = a + c$$

$$b = \frac{a + c}{2}.$$

That is, if three successive terms in a progression by difference be taken, the sum of the extremes is equal to twice the mean.

Example.

Let the three terms be 3, 5, and 7.

$$2 \times 5 = 7 + 3 = 10.$$

Example 2d. Let 7 and 17 be the first and last term, what is the mean?

$$x = \frac{7 + 17}{2} = 12.$$

Let a, b, c, d, be four successive terms of a progression by difference.

$$b - a = d - c$$
$$b + c = a + d.$$

That is, the sum of the two extremes is equal to the sum of the two means.

Example.

Let 5, 9, 13, 17, be four successive terms.

$$9 + 13 = 17 + 5 = 22.$$

Let $a, b, c, d, e \ldots\ldots h, i, k, l$, be any number of terms in a progression by differences; by the definition we have

$$b - a = c - b = d - c = e - d = i - h = k - i = l - k.$$
$$b - a = l - k$$
$$c - b = k - i$$
$$d - c = i - h, \text{ \&c.}$$

which by transposition give

$$a + l = b + k,$$
$$b + k = c + i.$$
$$c + i = d + h, \text{ \&c.}$$

That is, if the first and last be added together, the second and the last but one, the third and the last but two, the sums will all be equal.

Example,

Let 3, 5, 7, 9, 11, 13, be such a series.

$$\begin{array}{rrrrrr} 3, & 5, & 7, & 9, & 11, & 13, \\ 13, & 11, & 9, & 7, & 5, & 3, \\ \hline 16, & 16, & 16, & 16, & 16, & 16. \end{array}$$

It will now be easy to find the sum of all the terms in any progression by difference, and that even when but part of the terms are known.

Let S represent the sum of the series, then we have

$$S = a + b + c + d + \dots h + i + k + l.$$

Also $S = l + k + i + h + \dots d + c + b + a.$

Adding these term to term as they stand,

$$2S = (a + l) + (b + k) + (c + i) + (d + h) + \dots (d + h) + (c + i) + (b + k) + (a + l).$$

But it has just been shown that

$$a + l = b + k = c + i, \text{ \&c.}$$

That is, all the terms are now equal, and one of them being multiplied by the whole number of terms will give the whole sum : thus

$$2S = n(a + l)$$

$$S = \frac{n(n + l)}{2}$$

Hence, *the sum of a series of numbers in progression by difference is one half of the product of the number of terms by the sum of the first and last terms.*

Example.

How many strokes does the hammer of a clock strike in 12 hours?

$$a = 1, \quad l = 12, \quad \text{and } n = 12.$$

$$S = \frac{12\,(1 + 12)}{2} = 78. \quad \textit{Ans.}\ 78 \text{ strokes.}$$

In the formula $l = a + (n - 1)\,r$; substitute d instead of r to represent the difference; thus

$$l = a + (n - 1)\,d.$$

This formula and the following

$$S = \frac{n\,(a + l)}{2},$$

contain five different things, viz. a, l, n, d, and S; any three of which being given, the other two may be found, by combining the two equations. I shall leave the learner to trace these himself as occasion may require.

Examples in Progression by Difference.

1. How many strokes do the clocks of Venice, which go on to 24 o'clock, strike in a day?

2. Suppose 100 stones to be placed in a straight line 3 yards asunder; how far would a person travel who should set a basket 3 yards from the first, and then go and pick them up one by one, and put them into the basket?

3. After A, who travelled at the rate of 4 miles an hour, had been set out $2\frac{3}{4}$ hours, B set out to overtake him, and in order thereto went four miles and a half the first hour, four and three fourths the second, five the third, and so on, increasing his rate one fourth of a mile each hour. In how many hours will he overtake A?

The above example is solved by using both the above formulas. The known quantities are the first term, the difference, and the sum of all the terms. The unknown are the last term, and the number of terms. It involves an equation of the second degree. It is most convenient to use x, y, &c. for the unknown quantities.

4. A and B set out from London to go round the world, (24990 miles,) one going East and the other West. A goes one mile the first day, two the second, three the third, and so on, increasing his rate one mile per day. B goes 20 miles a day. In how many days will they meet, and how many miles will each travel?

5. A traveller sets out for a certain place, and travels 1 mile the first day, 2 the second, and so on. In 5 days afterwards another sets out, and travels 12 miles a day. How long and how far must he travel to overtake the first?

6. A and B 165 miles distant from each other set out with a design to meet; A travels 1 mile the first day, 2 the second, 3 the third, and so on. B travels 20 miles the first day; 18 the second, 16 the third, and so on. How soon will they meet?

Ans. They will be together on the 10th day, and continuing that rate of travelling, they may be together again on the 33d day. Let the learner explain how this can take place.

7. A person makes a mixture of 51 gallons, consisting of brandy, rum, and water; the quantities of which are in arithmetical progression. The number of gallons of brandy and rum together, is to the number of gallons of rum and water together as 8 to 9. Required the quantities of each.

Let $x =$ the number of gallons of rum

and $y =$ the common difference.

Then $x - y$, x, and $x + y$ will express the three quantities.

8. A number consisting of three digits which are in arithmetical progression, being divided by the sum of its digits, gives a quotient 48; and if 198 be subtracted from the number, the digits will be inverted. Required the number.

9. A person employed 3 workmen, whose daily wages were in arithmetical progression. The number of days they worked was equal to the number of shillings that the second received per day. The whole amount of their wages was 7 guineas, and the best workman received 28 shillings more than the worst. What were their daily wages?

Progression by difference is only a particular case of the series by difference, explained Arts. XL. and XLI. All the principles and rules of it may be derived from the formulas obtained there. It would be a good exercise for the learner to deduce these rules from those formulas.

XLVII. *Progression by Quotient, or Geometrical Progression.*

Progression by quotient is a series of numbers such, that if any term be divided by the one which precedes it, the quotient is the same in whatever part the two terms be taken. If the series is increasing, the quotient will be greater than unity, if decreasing, the quotient will be less than unity.

The following series are examples of this kind of progression.

$$3, \quad 6, \quad 12, \quad 24, \quad 48 \ldots\ldots \&c.$$
$$72, \quad 24, \quad 8, \quad \tfrac{8}{3}, \quad \tfrac{8}{9}, \quad \tfrac{8}{27}.$$

In the first the quotient (or *ratio*, as it is generally called,) is 2, in the second it is $\frac{1}{3}$.

Let $a, b, c, d, \ldots\ldots k, l,$ be a series of this kind, and let q represent the quotient.

Then we have by the definition,

$$q = \frac{b}{a} = \frac{c}{b} = \frac{d}{c} = \frac{e}{d} = \ldots\ldots \frac{l}{k}.$$

From these equations we derive

$$b = a\,q, \; c = b\,q, \; d = c\,q, \; e = d\,q \ldots\ldots l = k\,q.$$

Putting successively the value of b into that of c, and that of c into that of d, &c., they become

$$b = a\,q, \; c = a\,q^2, \; d = a\,q^3, \; c = a\,q^4, \ldots\ldots l = a\,q^{n-1},$$

designating by n, the rank of the term l, or the number of terms in the proposed progression.

Any term whatever in the series may be found without finding the intermediate terms, by the formula

$$l - a\,q^{n-1}.$$

Example.

What is the 7th term of the series 3, 6, 12, &c.?

Here $a = 3$, $q = 2$, and $n - 1 = 6$.

$$l = 3 \times 2^6 = 192.$$ *Ans.* 192.

We may also find the sum of any number of terms of the progression

$$a,\ b,\ c,\ d,\ \&c.$$

If we add the equations

$$b = a\,q,\ c = b\,q,\ d = c\,q,\ e = d\,q \ldots\ldots l = k\,q,$$

we obtain

$$b + c + d + e + \ldots\ldots l = (a + b + c + d + e + \ldots\ldots k)\,q.$$

Observe that the first member is the sum of all the terms of the progression except the first, a, and the part of the second member enclosed in the parenthesis, is the sum of all the terms except the last, l; and this, multiplied by q, is equal to the first member.

Now putting S for the sum of all the terms, we have

$$b + c + d + e + \ldots\ldots l = S - a$$
$$a + b + c + d + e + \ldots\ldots k = S - l.$$

Hence we conclude that

$$S - a = (S - l)\,q,$$

which gives

$$S = \frac{q\,l - a}{q - 1}.$$

Example.

What is the sum of seven terms of the series

$$5,\ 15,\ 45,\ \&c.$$

$$l = 5 \times 3^6 = 3645$$

$$S = \frac{3 \times 3645 - 5}{3 - 1} = 5465.$$

The two equations

$$l = a\,q^{n-1},\ \text{and}\ S = \frac{q\,l - a}{q - 1},$$

contain all the relations of the five quantities a, l, q, n, and S; any three of which being given, the other two may be found. It would however be difficult to find n, without the aid of logarithms, which will be explained hereafter. Indeed logarithms will greatly facilitate the calculations in most cases of geometrical progression. Therefore we shall give but few examples, until we have explained them.

If we substitute $a\,q^{n-1}$ in place of l, in the expression of S, it becomes

$$S = \frac{a\,(q^n - 1)}{q - 1}.$$

When q is greater than unity, the quantity q^n will become greater as n is made greater, and S may be made to exceed any quantity we please, by giving n a suitable value; that is, by taking a sufficient number of terms. But if q is a fraction less than unity, the greater the quantity n, the smaller will be the quantity q^n. Suppose $q = \frac{1}{m}$, m being a number greater than unity, then

$$q^n = \frac{1}{m^n}.$$

Substituting $\frac{1}{m^n}$ in place of q^n in the expression of S, and it becomes

$$S = \frac{a\left(\frac{1}{m^n} - 1\right)}{\frac{1}{m} - 1}.$$

Changing the signs of the numerator and denominator, and multiplying both by m,

$$S = \frac{a\,m\left(1 - \frac{1}{m^n}\right)}{m - 1} = \frac{a\,m - \frac{a\,m}{m^n}}{m - 1} = \frac{a\,m - \frac{a}{m^{n-1}}}{m - 1}.$$

It is evident that the larger n is or the more terms we take in the progression, the smaller will be the quantity $\frac{a}{m^{n-1}}$, and consequently the nearer the value of S will approach $\frac{am}{m-1}$, from which it differs only by the quantity

$$\frac{a}{(m-1)m^{n-1}}$$

But it can never, strictly speaking, be equal to it, for the quantity $\frac{a}{(m-1)m^{n-1}}$ will always have some value, however large n may be; yet no quantity can be assumed, but this expression may be rendered smaller than it.

The quantity $\frac{am}{m-1}$ is therefore the limit which the sum of a decreasing progression can never surpass, but to which the value continually approximates, as we take more terms in the series.

In the progression

$$1,\ \tfrac{1}{2},\ \tfrac{1}{4},\ \tfrac{1}{8},\ \tfrac{1}{16},\ \&c.$$

$$a = 1 \quad \frac{1}{m} = \tfrac{1}{2}.$$

$$\text{Hence } S = \frac{1 \times 2}{2-1} - \frac{1}{(2-1) \times 2^{n-1}} = 2 - \frac{1}{1 \times 2^{n-1}}.$$

In this example the more terms we take, the nearer the sum of the series will approach to 2, but it can never be strictly equal to it. Now if we consider the number of terms infinite, the quantity $\frac{1}{1 \times 2^{n-1}}$ will be so small that it may be omitted without any sensible error, and the sum of the series may be said to be equal to 2.

By taking more and more terms we approach 2 thus,

$$\begin{aligned}
&1 &&= 2 - 1\\
&1 + \tfrac{1}{2} &&= 2 - \tfrac{1}{2}\\
&1 + \tfrac{1}{2} + \tfrac{1}{4} &&= 2 - \tfrac{1}{4}\\
&1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} &&= 2 - \tfrac{1}{8}\\
&1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} &&= 2 - \tfrac{1}{16}, \text{ \&c.}
\end{aligned}$$

Examples.

What is the sum of the series 1, $\frac{1}{3}$, $\frac{1}{9}$, $\frac{1}{27}$, &c. continued to an infinite number of terms?

$$a = 1,\ \frac{1}{m} = \frac{1}{3}$$

$$S = \frac{1 \times 3}{3 - 1} = \frac{3}{2} = 1\tfrac{1}{2}.$$

2. What is the sum of the series, 5, $\frac{5}{3}$, $\frac{5}{9}$, $\frac{5}{27}$, &c. continued to an infinite number of terms?

3. What is the sum of the following series continued to infinity?

35, 7, $\frac{7}{5}$, $\frac{7}{25}$, &c.

4. What is the sum of the following series continued to infinity?

208, 26, $3\frac{1}{4}$, $\frac{13}{32}$, &c.

5. What is the sum of the following series continued to infinity?

38, $4\frac{2}{9}$, $1\frac{11}{27}$, $\frac{38}{81}$, &c.

6. What is the 10th term of the series

5, 15, 45, &c.?

7. What is the sum of 8 terms of the series

35, 175, 875, &c.?

When three numbers are in geometrical progression, the middle term is called a mean proportional between the other two.

Let three numbers, a, b, c, be in geometrical progression, so that

$$\frac{a}{b} = \frac{b}{c},$$

We have $$b^2 = a c$$

and $$b = (a c)^{\frac{1}{2}}.$$

8 Find a mean proportional between 4 and 9.

$$\frac{4}{x} = \frac{x}{9}$$

$$x^2 = 36$$

$$x = 6.$$ *Ans.* 6.

9. Find a mean proportional between 7 and 10.

10. Find a mean proportional between 2 and 3.

XLVIII. *Logarithms.*

We have seen, Art, XXXVIII, with what facility multiplication, division, the raising of powers, and the extraction of roots may be performed on literal quantities consisting of the same letter, by operating on the exponents. We propose now to apply the same principle, though in a way a little different, to numbers.

Multiplication, we observed, is performed by adding the exponents, and division by subtracting the exponent of the divisor from that of the dividend.

Thus $a^3 \times a^5$ is $a^{3+5} = a^8$. And $\frac{a^7}{a^5}$ is $a^{7-5} = a^2$.

In the same manner $2^3 \times 2^5 = 2^{3+5} = 2^8$,

and $$\frac{2^7}{2^5} = 2^{7-5} = 2^2.$$

Let us make a table consisting of two columns, the first containing the different powers of 2, and the second the exponents of those powers.

Observe first that $a^0 = 1$, so also $2^0 = 1$, $2^1 = 2$, $2^2 = 4$, $2^3 = 8$, $2^4 = 16$, $2^5 = 32$, $2^6 = 64$, $2^7 = 128$, &c.

TABLE.

Powers.	Expon.	Powers.	Expon.	Powers.	Expon.
1	0	128	7	16,384	14
2	1	256	8	32,768	15
4	2	512	9	65,536	16
8	3	1024	10	131,072	17
16	4	2048	11	262,144	18
32	5	4096	12	524,288	19
64	6	8192	13	1,048,576	20

Suppose now it is required to multiply 256 by 64. We find by the table that 256 is the 8th power of 2, that is 2^8, and that 64 is 2^6. Now $2^8 \times 2^6 = 2^{8+6} = 2^{14}$. Returning to the table again and looking for 14 in the column of exponents, against it we find 16384 for the 14th power of 2. Therefore the product of 256 by 64 is 16384.

This we may easily prove.

$$\begin{array}{r} 256 \\ 64 \\ \hline 1024 \\ 1536 \\ \hline 16384 \end{array}$$

Multiply 256 by 128.

Finding these numbers in the table in the column of powers, and looking in the other column for the exponents, we find that 256 is the 8th power of 2, and 128 the 7th power. Adding the exponents 8 and 7, we have 15 for the exponent of the product. Now looking for 15 in the column of exponents, we find against it in the column of powers, 32768 for the 15th power of 2, which is the product of 256 by 128. Let the learner prove this by multiplying 256 by 128.

Divide 8192 by 32.

Looking for these numbers in the column of powers, and for the corresponding exponents, we find 8192 is the 13th power of 2, and 32 is the 5th power.

$$\frac{2^{13}}{2^5} = 2^{13-5} = 2^8.$$

Looking for 8 in the column of exponents, and for its corresponding number, we find 256 for the 8th power of 2, or the quotient of 8192 by 32.

Divide 32768 by 512.

The exponents corresponding to these numbers in the table are 15 and 9. $15 - 9 = 6$. In the column of exponents, 6 corresponds to 64, which is the true quotient of 32768 by 512

What is the 3d power of 32?

The exponent corresponding to 32 is 5. Now to find the 3d power of a^5 we should multiply the exponent by 3, thus $a^{5 \times 3} = a^{15}$. So the third power of 2^5 is $2^{5 \times 3} = 2^{15}$. Against 15 in the column of exponents we find 32768 for the 15th power of 2. Therefore the 3d power of 32 is 32768.

What is the 2d power of 128?

The exponent corresponding to this number is 7. $7 \times 2 = 14$. The number corresponding to the exponent 14 is 16384, which is the second power of 128.

What is the 3d root of 4096?

The exponent corresponding to this number is 12.

The 3d root of 2^{12} is $2^{\frac{12}{3}} = 2^4$.

The number corresponding to the exponent 4 is 16, which is the 3d root of 4096.

What is the fourth root of 65,536?

The exponent corresponding to this number is 16, which divided by 4 gives for the exponent of the root 4, the number corresponding to which is 16. The answer is 16.

Examples.

1. Multiply 512 by 256.
2. Multiply 8192 by 128.
3. Multiply 2048 by 256.
4. Divide 262,144 by 128.
5. Divide 1,048,576 by 512.
6. Divide 524,288 by 131,072.
7. What is the 2d power of 1024?
8. What is the 3d power of 64?
9. What is the 5th power of 16?
10. What is the 2nd root of 262,144?
11. What is the 3d root of 262,144?
12. What is the 4th root of 1,048,576?
13. What is the 5th root of 1,048,576?
14. What is the 6th root of 262,144?

The operations of multiplication, division, and the extraction of roots are very easy by means of this table. This table however contains but very few numbers. But an exponent of 2 may be found for all numbers from 1 as high as we please. For $2^1 = 2$, and $2^2 = 4$. Hence the exponent of 2 answering to the number 3 will be between 1 and 2; that is, 1 and a fraction. So the exponents answering to 5, 6, and 7, will be 2 and a fraction, &c.

XLIX. A table may also be made of the powers of 3, or of 4, or any other number except 1, which shall have the same properties. Exponents might be found answering to every number from 1 upwards.

$$3^0 = 1,\ 3^1 = 3,\ 3^2 = 9,\ 3^3 = 27, \&c.$$

The column of powers will always consist of the numbers 1, 2, 3, &c. but the column of exponents will be different according as the numbers are considered powers of a different number.

The formula $a^x = y$ will apply to every table of this kind.

If any number except 1 be put in the place of a, and y be made successively 1, 2, 3, 4, a suitable value may be found for x, which shall answer the conditions.

If a be made 1, y will always be 1, whatever value be given to x; for all powers, as well as all roots of 1, are 1.

But if any number greater than 1 be put in the place of a, y may equal any number whatever, by giving x a suitable value.

Giving a value to a then, we begin and make y successively 1, 2, 3, 4, &c. and these numbers will form the first column or columns of powers in the table. Then we find the values of x corresponding to these values of y, and write them in the second column against the values of y, and these form the column of exponents. These exponents are called *logarithms.* The first column is usually called the column of numbers, and the second, the column of logarithms. The number put in the place of a, is called the base of the table. Whatever number is made base at first, must be continued through the table.

Observe that $a^0 = 1$; therefore whatever base be used, the logarithm of 1 is zero. And 1 will be the logarithm of the base, for $a^1 = a$.

The most convenient number for the base, and the one generally used in the tables, is 10.

$10^0 = 1$, $10^1 = 10$, $10^2 = 100$, $10^3 = 1000$, $10^4 = 10000$, $10^5 = 100000$, $10^6 = 1000000$, &c.

Now to find the logarithm of 2, 3, 4, &c.

Make $10^x = 2$, $10^{x'} = 3$, $10^{x''} = 4$, &c.

For all numbers between 1 and 10, x must be a fraction, because $10^0 = 1$ and $10^1 = 10$.

Make $x = \frac{1}{z}$, then it becomes

$$10^{\frac{1}{z}} = 2.$$

As the process for finding the value of z in this equation is long and rather too difficult for young learners, we will suppose it already found.

$$\frac{1}{z} = .30103 \text{ very nearly.}$$

Hence $10^{\frac{30103}{100000}} = 2$ very nearly.

To understand this, we must suppose 10 raised to the 30103d power, and then the 100000th root of it taken, and this will differ very little from 2. The number .30103 is the logarithm of 2. The fractional part of logarithms is always expressed in decimals.

Having the logarithm of 2, we may find the logarithm of 4 by doubling it, for $2^2 = 4$. That of $8 = 2^3$ is found by tripling it, and so on.

The logarithm of 4 is $.30103 \times 2 = .60206$.

The logarithm of 8 is $.30103 \times 3 = .90309$.

The logarithm of 16 is $.30103 \times 4 = 1.20412$, &c.

Again $10^{\frac{4771213}{10000000}} = 3$ very nearly.

Hence the logarithm of 3 is .4771213.

Since $2 \times 3 = 6$, the logarithm of 6 is found by adding the logarithm of 2 and 3 together.

$.30103 + .4771213 = .7781513 =$ logarithm of 6.

Since $3^2 = 9$, the logarithm of 9 is found by multiplying that of 3 by 2. With the logarithms of 2 and 3 the logarithms of all the powers of each, and of all the multiples of the two may be found.

The logarithm of 5 may be found by subtracting that of 2 from that of 10, since $5 = \frac{10}{2}$. The logarithm of 10 is 1.

$$1 - .30103 = .69897 = \text{log. of } 5.$$

Now all the logarithms of all the multiples of 2, 3, 5, and 10 may be found. Hence it appears that it is necessary to find the logarithms of the prime numbers, or such as have no divisor except unity, by trial; and then the logarithms of all the compound numbers may be found from them.

The decimal parts of the logarithms of 20, 30, &c. are the same as those of 2, 3, 4, &c. For, since the logarithm of 10 is 1; that of 100, 2; that of 1000, 3, &c., it is evident that adding these logarithms to the logarithms of any other numbers, will not alter the decimal part. Hence 1 added to the logarithm of 2 forms that of 20, and 2 added to the logarithm of 2 forms that of 200, &c.

Log. 2 = .30103, log. 20 = 1.30103, log. 200 = 2.30103, log. 2000 = 3.30103.

The logarithm of 25 is 1.39794; that of $250 = 25 \times 10$ is $1 + 1.39794 = 2.39794$; that of $2500 = 25 \times 100$ is $2 + 1.39794 = 3.39794$.

The logarithms of all numbers below 10 are fractions, those of all the numbers between 10 and 100 are 1 and a fraction; those of all numbers between 100 and 1000 are 2 and a fraction; those of all numbers between 1000 and 10000 are 3 and a fraction. That is, the whole number which precedes the fraction in the logarithm is always equal to the number of figures in the number less one. This whole number is called the *index* or *characteristic* of the logarithm. Thus in the logarithm 2.3576423, the figure 2 is the characteristic showing that it is the logarithm of a number consisting of three figures or between 100 and 1000.

As the characteristic may always be known by the number, and the number of figures in a number may be known by the characteristic, it is usual to omit the characteristic in the table, to save the room. It is useful to omit it too, because the same fractional part, with different characteristics, forms the logarithms of several different numbers.

The logarithm of 37 is 1.568202.

$$\frac{37}{10} = 3.7 = \frac{10^{1.568202}}{10} = 10^{.568202}.$$

The logarithm of 3.7 is .568202, which is the same as that of 37, with the exception of the index.

$$\frac{3762}{100} = 37.62 = \frac{10^{3.575419}}{10^{2}} = 10^{1.575419}.$$

$$\frac{3762}{1000} = 3.762 = \frac{10^{3.575419}}{10^{3}} = 10^{.575419}.$$

That is, all numbers which are tenfold, the one of the other, have the same logarithm.

376200	has for its logarithm	5.575419.
37620	“ “	4.575419.
3762	“ “	3.575419.
376.2	“	2.575419.
37.62	“	1.575419.
3.762	“	0.575419.

When a number consists of whole numbers and decimal parts, we find the fractional part of the logarithm in the same manner as if all the figures of the number belonged to the whole number, but we give it the index corresponding to the whole number only.

In most tables of logarithms they are carried as far as seven decimal places. Some however are only carried to five or six. The disposition of the tables is something different in different sets, but they are generally accompanied with an explanation. When one set of tables is well understood, all others will be easily learned. The logarithms for the following examples may be found in any table of logarithms. They are used here as far as six places.

Examples.

1. Multiply 43 by 25.

Find 43 in the column of numbers, and against it in the column of logarithms you will find 1.633468, and against 25 you will find 1.397940. Add these two logarithms together and their sum is the logarithm of the product.

log.	43	1.633468
“	25	1.397940
“	1075	3.031408

Find this logarithm in the column of logarithms, and against it in the column of numbers you find 1075 which is the product of 43 multiplied by 25. The index, 3, shows that the number must consist of four places.

Let the learner prove the results at first by actual multiplication.

2. Multiply 2520 by 300.

By what was remarked above, the logarithm of 2520 is the same as that of 252 with the exception of the index, and that of 300 is the same as that of 3 except the index.

Find the number 252 in the left hand column, and against it in the second column you find .401401. The number 2520 consists of four places, therefore the index of its logarithm must be (4 — 1) or 3. The logarithm corresponding to 300 is .477121, and its index must be 2, because 300 consists of three places.

log.	2520	3.401401
"	300	2.477121
"	756000	5.878522

Find this logarithm, and against it in the column of numbers you will find 756; but the index 5 shows that the number must consist of 6 places; therefore three zeros must be annexed to the right, which makes the number 756000, which is the product of 2520 by 300.

3. Multiply 2756 by 20.

* To find the logarithm of 2756, find in the column of numbers 275, and at the top of the table look for 6. In the column under 6 and opposite 275 you find .440279 for the decimal part of the logarithm of 2756. The characteristic will be 3.

log.	2756	3.440279
"	20	1.301030
"	55120	4.741309

Looking in the table for this logarithm, against 551 you will find .741152 and against 552 you will find .741939. The logarithm .741309 is between these two. Against 551, look along in the other columns. In the column under 2 you find the logarithm required. The figures of the number, then, are

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* In some tables the whole number 2756 may be found in the left hand column.

5512, but the characteristic being 4, the number must consist of five places; hence annexing a zero, you have 55120 for the product of 2756 by 20.

4. Divide 756342 by 27867.

Both these numbers exceed the numbers in the tables, still we shall be able to find them with great accuracy. First find the logarithm of 756300, which is 5.878694. The difference between this logarithm and that of 756400 is 58. The difference between 756300 and 756400 is 100, and the difference between 756300 and 756342 is 42. Therefore, if $\frac{42}{100} = .42$ of 58 be added to the logarithm of 756300, it will give the logarithm of 756342 sufficiently exact, $58 \times .42 = 24$, rejecting the decimals. $5.878694 + 24 = 5.878718$. The 58, and consequently the 24, are decimals of the order of the two last places of the logarithm, but this circumstance need not be regarded in taking these parts. It is sufficient to add them to their proper place.

The table generally furnishes means of taking out this logarithm more easily. As the differences do not often vary an unit for considerable distance among the higher numbers, the difference is divided into ten equal parts, (that is, as equal as possible, the nearest number being used, rejecting the decimal parts) and one part is set against 1, two parts against 2, &c. in a column at the right of the table.

In the present case, then, for the 4 (for which we are to take $\frac{4}{10}$ of 58,) we look at these parts and against it we find 23, and for the 2 (for which we must take $\frac{2}{100}$ of 58,) we find 11. But 11 is $\frac{2}{10}$, consequently to obtain $\frac{2}{100}$ we must take $\frac{1}{10}$ of 11 which is 1, omitting the decimal. The operation may stand thus:

log. 756300	5.878694
$\frac{4}{10}$ of diff.	23
$\frac{2}{100}$	1
log. 756342	5.878718

To find the logarithm of 27867, proceed in the same manner, first finding that of 27860, and then adding $\frac{7}{10}$ of the difference, which will be found at the right hand, as above.

log. 27860	4.444981
$\frac{7}{10}$ diff.	109
log. 27867	4.445090
From log. dividend	5.878718
Subt. log. of divisor	4.445090
log. of quotient 27.141	1.433628

We find the decimal part of this logaritnm is between .433610 and .433770, the former of which belongs to the number 2714, and the latter to 2715. Subtract 433610 from 433628, the remainder is 18. Looking in the column of parts, the number next below 18 is 17, which stands against 1 or $\frac{1}{10}$ of the whole difference.

Put this 1 at the right of 2714, which makes 27141. The characteristic 1 shows that the number is between 10 and 100. Therefore the quotient is 27.141. This quotient is correct to three decimal places.

If the table has no column of differences, take the whole difference between .433610 and .433770, which is 160 for a divisor, the 18 for a dividend, annexing one or more zeros. One place must be given to the quotient for each zero.

$$\begin{array}{r|l} 180 & 160 \\ \cline{2-2} 160 & .1 \end{array}$$

5. What is the 3d power of 25.7 ?

log. 25.7		1.409933
Multiply by 3		3
log.	16974.6	4.229799
	Ans.	16974.6—.

6. What is the 3d root of 15 ?

log. 15	1.176091 (3
log. 2.46621	.392030
Ans.	2.4662+.

L. Since a fraction consists of two numbers, one for the numerator and the other for the denominator, the logarithm of a fraction must consist of two logarithms; and as a fraction expresses the division of the numerator by the denominator, to express this operation on the logarithms, that of the denominator must be represented as to be subtracted from the numerator.

The logarithm of $\frac{3}{5}$ is expressed thus:

$$\text{log. } 3 - \text{log. } 5 = 0.477121 - 0.698970.$$

The logarithm of a fraction whose numerator is 1, may be expressed by a single logarithm. For $\frac{1}{a^2}$ is the same as a^{-2}. If we would express the logarithm of $\frac{1}{3}$, for example,

$$10^{.477121} = 3, \text{ consequently } \frac{1}{10^{.477121}} = 10^{-.477121} = \tfrac{1}{3}.$$

That is, the logarithm of $\frac{1}{3}$ is the same as the logarithm of 3, except the sign, which for the fraction is negative. Any fraction may be reduced to the form $\frac{1}{a^n}$, but the denominator will consist of decimals or still contain a fraction.

$$\tfrac{3}{5} = \frac{1}{1\frac{2}{3}} = \frac{1}{1.666 +}.$$

If the subtraction be actually performed, on the expression of this fraction given above, it will be reduced to the logarithm of a fraction of this form.

$$0.477121 - 0.698970 = -0.221849.$$

The number corresponding to the logarithm 0.221849 is 1.666 +, but the sign being negative, shows that the number is

$$\frac{1}{1.666 +}.$$

The logarithms of all common fractions may be obtained in either of the above forms, but they are extremely inconvenient in practice. The first on account of its consisting of two logarithms would be useless as well as inconvenient; because though we might find a logarithm corresponding to any fraction, yet in performing operations, a logarithm would never be found in that form when it was required to find its number.

The second form is inconvenient because it is negative, and also because in seeking the number corresponding to the logarithm, a fraction would frequently be found with decimals in the denominator. It would be much better that the whole fraction should be expressed in decimals. If the fraction is used in the decimal form, the logarithms may be used for them almost as easily as for whole numbers.

Suppose it is required to find the logarithm of .5 or $\frac{5}{10}$.

$$\text{log. } 5 - \text{log. } 10 = 0.698970 - 1. = -1 + .698970.$$

Suppose it is required to find the logarithm of .05 or $\frac{5}{100}$.

$$\frac{5}{100} = \frac{10^{.698970}}{10^2} = 10^{-2+.698970.}$$

$$\text{log. } 5 - \text{log. } 100 = 0.698970 - 2 = -2 + .698970.$$

The logarithms of 10, 100, 1000, &c. always being whole numbers, we have the two parts distinct. The logarithm of .5 is the same as that of 5 except that it has the number 1 joined to it with the sign —, which is sufficient to distinguish it, and show it to be a fraction. The logarithm of .05 also is the same, except that — 2 is joined to it. That is, the logarithm of the numerator is positive, and that of the denominator negative.

This negative number joined to the positive fractional part, serves as a characteristic, and is a continuation of the principle shown above; thus

The	log.	500	is	2.698970
“	“	50		1.698970
“	“	5		0.698970
“	“	.5		$\overline{1}.698970$
“	“	.05		$\overline{2}.698970$

The logarithm of a decimal is the same as that of a whole number expressed by the same figures, with the exception of the characteristic, which is negative for the fraction; being — 1 when the first figure on the left is tenths, — 2 when the first is hundredths, &c. It is convenient to write the sign over the characteristic thus, $\overline{1}$, $\overline{2}$, &c. It is not necessary to put the sign + before the fractional part, for this will always be understood to be positive.

In operating upon these numbers, the same rules must be observed as in other cases where numbers are found connected with the signs + and —.

When the first figure of the fraction is tenths, the characteristic is $\bar{1}$, when the first is hundredths, the characteristic is $\bar{2}$, &c.

The log. of .25 is log. 25 — log. 100

$$= 1.397940 - 2 = -2 + 1.397940 = \bar{1}.397940.$$

This is the same as the logarithm of 25, except that the characteristic $\bar{1}$ shows that its first figure on the left is 10ths, or one place to the right of units.

Multiply 325 by .23.

log. 325 . . . . 2.511883
log. .23 . . . $\bar{1}.361728$

log. 74.75 *Ans.* . . 1.873611

Multiply 872 by .097.

log. 872 . . . 2.940516
log. .097 . . . $\bar{2}.986772$

log. 84.584 *Ans.* . . 1.927288

In adding the logarithms, there is 1 to carry from the decimal to the units. This one is positive, because the decimal part is so.

Multiply .857 by .0093

log. .857 . . . $\bar{1}.932981$
log. .0093 . . . $\bar{3}.968483$

log. .0079701 *Ans.* . . $\bar{3}.901464$

Divide 75 by .025.

log. 75 . . . 1.875061
log. .025 . . $\bar{2}.397940$

log. 3000 *Ans.* . . . 2.477121

In subtracting, the negative quantity is to be added, as in algebraic quantities.

Divide 275 by .047.

log. 275 . . . 2.439333
log. .047 . . $\bar{2}.672098$

log. 5851.07 *Ans.* . . . 3.767235

Divide .076 by 830.

log. .076 $\bar{2}.880814 = \bar{3} + 1.880814$
log. 830 . . 2.919078

log. .0000915662 *Ans.* . $\bar{5}.961736$

In order to be able to take the second from the first, I change the characteristic $\bar{2}$ into $\bar{3} + 1$ which has the same value. This enables me to take 9 from 18, that is, it furnishes a ten to borrow for the last subtraction of the positive part. In subtracting, the characteristic 2 of the second logarithm becomes negative and of course must be added to the other negative.

Divide .735 by .038.

log. .735 . . $\bar{1}.866287$
log. .038 . . $\bar{2}.579784$

log. 19.3422 *Ans.* . . 1.286503

What is the 3d power of .25?

log. .25 . . $\bar{1}.397940$
3

log. 0.015625 *Ans.* $\bar{3} + 1.193820 = \bar{2}.193820.$

What is the third root of 0.015625?

The logarithm of this number is $\bar{2}.193820$. This characteristic $\bar{2}$ cannot be divided by 3, neither can it be joined with the first decimal figure in the logarithm, because of the different sign. But if we observe the operation above in finding the power, we shall see, that in multiplying the decimal part there was 1 to carry, which was positive, and after the multiplication was completed, the characteristic stood thus, $\bar{3} + 1$ which was afterwards reduced to $\bar{2}$. Now if we add $\bar{1}+1$ to the $\bar{2}$ in the present instance, it will become $\bar{3}+1$, and at the same time its value will not be altered. The negative part of the characteristic will then be divisible by 3, and the 1 being positive may be joined to the fractional part.

log. .015625 $\quad \bar{2}.193820 = \bar{3} + 1.193820(3$

log. .25 *Ans.* $\quad \bar{1}.397940$

In all cases of extracting roots of fractions, if the negative characteristic is not divisible by the number expressing the root, it must be made so in a similar manner.

If the characteristic were $\bar{3}$ and it were required to find the fifth root, we must add $\bar{2}+2$ and it will become $\bar{5}+2$.

What is the 4th root of .357?

log. .357 $\quad \bar{1}.552668 = \bar{4} + 3.552668(4$

log. .77294 *Ans.* $\quad \bar{1}.888147$

Any common fraction may be changed to a decimal by its logarithms, so that when the logarithm of a common fraction is required, it is not necessary to change the fraction to a decimal previous to taking it.

It is required to find the logarithm of ½ corresponding to ½ expressed in decimals.

The logarithm of 2 being 0.30103, that of ½ will be —0.30103.

Now $\quad -0.30103 = -1 + 1 - .30103$

$= -1 + (1 - .30103) = \bar{1}.69897.$

The decimal part .69897 is the log. of 5, and — 1 is the log. of 10 as a denominator. Therefore $\bar{1}.69897$ is the log. of .5 = ½.

Reduce ⅝ to a decimal.

log. 5 $\quad 0.69897 = -1 + 1.698970$

log. 8 $\quad 0.903090$

log. 0.625 = ⅝ *Ans.* $\quad \bar{1}.795880$

When there are several multiplications and divisions to be performed together, it is rather more convenient to perform the whole by multiplication, that is, by adding the logarithms. This may be effected on the following principle. To divide by 2 is the same as to multiply by $\frac{1}{2}$ or .5. Dividing by 5 is the same as multiplying by $\frac{1}{5}$ or .2, &c.

Suppose then it is required to divide 435 by 15. Instead of dividing by 15 let us propose to multiply by $\frac{1}{15}$. First find the logarithm of $\frac{1}{15}$ reduced to a decimal.

log. 1		is $0 = -2 + 2.000000$
log. 15	subtract	1.176091
log. $\frac{1}{15}$ in form of a decimal		$\overline{2}.823909$
log. 435	add	2.638489
log. 29 = quotient of 435 by 15		1.462398

The log. of $\frac{1}{15}$ viz. $\overline{2}.823909$ is called the *Arithmetic Complement* of the log. of 15.

The arithmetic complement is found by subtracting the logarithm of the number from the logarithm of 1, *which is zero, but which may always be represented by* $\overline{1} + 1$, $\overline{2} + 2$, *&c. It must always be represented by such a number that the logarithm of the number may be subtracted from the positive part. That is, it must always be equal to the characteristic of the logarithm to be subtracted, plus* 1; *for* 1 *must always be borrowed from it, from which to subtract the fractional part.*

It is required to find the value of x in the following equation.

$$x = \left(\frac{35 \times 28 \times 56.78}{387 \times 2.896}\right)^{\frac{3}{5}}$$

log. 35			1.544068
log. 28			1.447158
log. 56.78			1.754195
log. 387	2.587711	Arith. Com.	$\bar{3}$.412289
log. 2.896	0.461799	" "	$\bar{1}$.538201
			1.695911
			3
			5.087733(5
log. 10.4123 very nearly answer			1.017546

I multiply by 3 to find the 3d power, and divide by 5 to obtain the 5th root.

LI. There is an expedient generally adopted to avoid the negative characteristics in the logarithms of decimals. I shall explain it and leave the learner to use the method he likes the best.

1. Multiply 253 by .37.

log. .37	$\bar{1}$.568202
log. 253	2.403121
log. 93.61 nearly answer	1.971323

Instead of using the logarithm $\bar{1}$.568202 in its present form, add 10 to its characteristic and it becomes 9.568202.

log. .37	9.568202
log. 253	2.403121
	11.971323
Subtract	10.
log. 93.61 as above.	1.971323

In this case 10 was added to one of the numbers and afterwards subtracted from the result; of course the answer must be the same.

2. Multiply .023 by .976.

Take out the logarithms of these numbers and add 10 to each characteristic.

log. .023	8.361728
log. .976	9.989450
	18.351178
Subtract	20
log. .0224473 nearly ans.	$\bar{2}$.351178

We may observe that, in this way, when the first left hand figure is tenths, the characteristic, instead of being $\bar{1}$ is 9, and when the first figure is hundredths, the characteristic is 8, &c. That is, the place of the first figure of the number reckoned from the decimal point corresponds to what the characteristic falls short of 10. Whenever in adding, the characteristic exceeds 10, the ten or tens may be omitted and the unit figure only retained.

In the first example, one number only was a fraction, viz. .37. In adding, the characteristic became 11, and omitting the 10 it became 1, which shows that the product is a number exceeding 10.

In the second example both numbers were fractions, of course each characteristic was 10 too large. In adding, the characteristic became 18. Now instead of subtracting both tens or 20, it is sufficient to subtract one of them, and the characteristic 8, which is 2 less than 10, shows as well as $\bar{2}$ would do, that the product is a fraction, and that its first figure must be in the second place of fractions or hundredth's place.

If three fractions were to be multiplied together, there would be three tens too much used, and the characteristic would be between 20 and 30; but rejecting two of the tens, or 20, the remaining figure would show the product to be a fraction, and show the place of its first figure.

3. What is the 3d power of .378?

log. .378	9.577492
Multiply by	3
	28.732476
log. .05401 nearly ans.	8.732476

Multiplying by 3 is the same as adding the number twice to itself. The characteristic becomes 28, but omitting two of the tens or 20, it becomes 8, which shows it to be the logarithm of a fraction whose first place is hundredths.

If it is required to find the 3d root of a fraction, it is easy to see, that having taken out the logarithm of the fraction, it will be necessary to add two tens to the characteristic, for it is then considered the third power of some other fraction, and in raising the fraction to that power, two tens would be subtracted.

In the last example the logarithm of the power is 8.732476; but in order to take its 3d root, it will be necessary to add the two tens which were omitted.

For the second root one ten must be previously added, and for the fourth root, three tens, &c.

4. What is the 3d root of .027 ?

log. .027	8.431364
or considered as a 3d power	28.431364 (3
log. .3. *Ans.* . . .	9.477121

5. What is the 2d root of .0016 ?

log. .0016 . .	7.204120
or considered a second power	17.204120 (2
log. .04. *Ans.*	8.602060

In dividing a whole number by a fraction, if 10 be added to the characteristic of the dividend, it cancels the 10 supposed to be added to the divisor. If both are fractions the ten in the one cancels it in the other; and if the dividend only is a fraction, the answer will of course be a less fraction. Consequently in division the results will require no alteration.

6. Divide 57 by .018.

log. 57 . .	1.755875
log. .018 . .	8.255272
log. 3166.7 nearly ans. .	3.500603

Here in subtracting I suppose 10 to be added to the first characteristic, and say 8 from 11, &c.

7. Divide .2172 by .006.

log. .2172	. . . .	9.336860
log. .006	. . .	7.778151
log. 36.2 *Ans.*	. .	1.558709

In taking the arithmetical complement, the logarithm of the number may be subtracted immediately from 10. The logarithm of 2 being .301030, its arithmetical complement is $\bar{1}.698970$. Adding 10 it becomes 9.698970. It would be the same if subtracted immediately from 10 thus 10 — .301030 = 9.698970.

8. It is required to find the value of x in the following expression:

$$x = \frac{17}{112}\left(\frac{13.73 \times .0706}{.253}\right)^{\frac{3}{2}}$$

log. 13.73			1.137670
log. .0706			8.848805
log. .253	9.403121	Arith. Com.	0.596879
Sum			0.583354
			3
Product by 3			1.750062 (2
Quotient by 2			0.875031
log. 17			1.230449
log. 112	2.049218	Arith. Com.	7.950782
log. x =	1.13836 nearly		0.056262

Find the value of x in the following equations.

9. $$x = \left(\frac{38.47 \times .463}{.637 \times 576}\right)^{\frac{2}{3}}$$

10. $$x = \frac{345}{417} \cdot \left(\frac{872 \times .0065}{.038 \times 4685}\right)^{\frac{3}{5}}$$

11. $$x = \frac{25}{476} \cdot \left(\frac{873}{956}\right)^{3} \cdot \left(\frac{278}{1973}\right)^{\frac{3}{4}}$$

12. $$38^{x} = 583.$$

Observe that the 2d power of 38 is found by multiplying the logarithm of 38 by 2, the 3d power by multiplying it by 3, &c. which will give the logarithm of the result. Hence we have the following equation; the logarithm of 38 being 1.579784 and that of 583 being 2.765669.

$$x \times 1.579784 = 2.765669$$

$$x = \frac{2.765669}{1.579784} = 1.75066 +$$

The value of x is found by dividing one logarithm by the other in the same manner as other numbers. It might be done by logarithms if the tables were sufficiently extensive to take out the numbers. By a table with six places an answer correct to four decimal places may be obtained.

In taking out the logarithms the right hand figure may be omitted without affecting the result in the first four decimals.

log. 2.76567 . .	0.441794
log. 1.57978 . . .	0.198588
log. $x = 1.75064 +$.	0.243206

13. What is the value of x in the equation $1537^{\frac{1}{x}} = 52$?

This gives first $1537 = 52^{x}$.

This may now be solved like the last.

LII. *Questions relating to Compound Interest.*

It is required to find what any given principal p will amount to in a number n of years, at a given rate per cent. r, at compound interest.

Suppose first, that the principal is \$1, or £1, or one unit of money of any kind.

The interest of 1 for one year is $\frac{1 \times r}{100}$, or simply r, if r is considered a decimal. The amount of 1 for one year then, will be $1 + r$. The amount of p dollars will be $p\,(1 + r)$.

For the second year, $p\,(1 + r)$ will be the principal, and the amount of 1 being $(1 + r)$, the amount of $p\,(1 + r)$ will be $p\,(1 + r)\,(1 + r)$ or $p\,(1 + r)^2$.

For the third year $p\,(1 + r)^2$ being the principal, the amount will be $p\,(1 + r)^2\,(1 + r)$ or $p\,(1 + r)^3$.

For n years then, the amount will be $p\,(1 + r)^n$.

Putting A for the amount, we have

$$A = p\,(1 + r)^n.$$

This equation contains four quantities, A, p, r, and n, any three of which being given, the other may be found.

Logarithms will save much labour in calculations of this kind.

Examples.

1. What will \$753.37 amount to in $5\frac{3}{8}$ years, at 6 per cent. compound interest?

Here $p = 753.37$, $r = .06$, and $n = 5\frac{3}{8}$.

$$\begin{array}{llr}
\log.\ 1 + r = 1.06 & \quad . \quad . & 0.025306 \\
 & n = & 5\frac{3}{8} \\ \hline
 & & 0.003163\frac{1}{4} \\
 & & 3 \\ \hline
 & & 0.009490 \\
 & & 0.126530 \\ \hline
\log.\ (1 + r)^{5\frac{3}{8}} & \quad . \quad . \quad . \quad . & 0.136020 \\
\log.\ 753.37 & \quad . \quad . \quad . & 2.877008 \\ \hline
\log.\ \$1030.457\ \textit{Ans.} & & 3.013028
\end{array}$$

2. What principal put at interest will amount to $5000 in 13 years at 5 per cent. compound interest?

By the above formula

$$p = \frac{A}{(1+r)^n}$$

$$\begin{array}{lrr}
\text{log. } 1 + r = 1.05 & & 0.021189 \\
 & n = & 13 \\
\hline
 & & .063567 \\
 & & 21189 \\
\hline
 & \text{Subtract} & .275457 \\
\text{log. } A = 5000 & \text{From} & 3.698970 \\
\hline
\text{log. } p = \$2651.60 \text{ nearly } \textit{Ans.} & & 3.423513
\end{array}$$

3. At what rate per cent. must $378.57 be put at compound interest, that it may amount to $500 in 5 years?

Solving the equation $A = p\ (1 + r)^n$ making r the unknown quantity, it becomes

$$r + 1 = \left(\frac{A}{p}\right)^{\frac{1}{n}}$$

$$\begin{array}{lr}
\text{log. } A = 500 \quad . \quad . \quad . \quad . & 2.698970 \\
\text{log. } p = 378.57 \quad . \quad . \quad . \quad . & 2.578146 \\
\hline
\quad \text{Dividing by } n = 5 & 0.120824 \ (5 \\
\hline
\text{log. } (r + 1) = 1.05722 & 0.024165
\end{array}$$

Consequently $r = 0.05722$ *Ans.*

4. In what time will $284.37 amount to 750 at 7 per cent.?

Making n the unknown quantity, the equation $A = p\,(1 + r)^n$ becomes

$$\text{log. } \frac{A}{p} = n \times \text{log. } (1 + r), \text{ and}$$

$$n = \frac{\log.\left(\frac{A}{p}\right)}{\log.(1+r)}$$

log. $A = 750$	2.875061
log. $p = 284.37$	2.453881
log. $\frac{A}{p}$	0.421180
log. $1 + r = 1.07$, is 0.029384	
log. 0.421180	9.624467
log. 0.029384	8.468111
log. $n = 14.334$ nearly *Ans.*	1.156356

5. What will be the compound interest of $947 for 4 years and 3 months at 5½ per cent.?

6. What will $157.63 amount to in 17 years at 4¾ per cent.?

7. A note was given the 15th of March 1804, for $58.46, at the rate of 6 per cent. compound interest; and it was paid the 19th of Oct. 1823. To how much had it amounted?

8. A note was given the 13th of Nov. 1807, for $456.33, and was paid the 23d of Sept. 1819. The sum paid was $894.40. What per cent. was allowed at compound interest?

9. In what time will the principal p be doubled, or become $2p$, at 6 per cent. compound interest? In what time will it be tripled?

Note. In order to solve the above question, put $2p$ in the place of A for the first, $3p$ for the second, and find the value of n.

The principles of compound interest will apply to the following questions concerning the increase of population.

10. The number of the inhabitants of the United States in A. D. 1790 was 3,929,000, and in 1800, 5,306,000. What rate per cent. for the whole time was the increase? What per cent. per year?

11. Suppose the rate of increase to remain the same for the next 10 years, what would be the number of inhabitants in 1810?

12. At the same rate, in what time would the number of inhabitants be doubled after 1800?

13. The number of inhabitants in 1810 by the census was 7,240,000. What was the annual rate of increase?

14. At the above rate, what would be the number in 1820?

15. At the above rate, in what time would the number in 1810 be doubled?

16. The number of inhabitants by the census of 1820, was 9,638,000. What was the annual rate of increase from 1810 to 1820?

17. At the same rate, what is the number in 1825?

18. At the same rate, what will be the number in 1830?

19. At the same rate, in what time will the number in 1820 be doubled?

20. In what time will the number in 1820 be tripled?

21. When will the number of inhabitants, by the rate of the last census, be 50,000,000?

LIII. 1. Suppose a man puts $10 a year into the savings bank for 15 years, and that the rate of interest which the bank is able to divide annually is 5 per cent. How much money will he have in the bank at the end of the 15th year?

Suppose a = the sum put in annually,
r = the rate of interest,
t = the time,
A = the amount.

According to the above rule of compound interest, the sum a at first deposited will amount to $a\ (r + 1)^t$; that deposited the second year will amount to $a\ (r + 1)^{t-1}$; that deposited the third year will amount to $a\,(r + 1)^{t-2}$; that deposited the last year will amount to $a\,(r + 1)^1$. Hence we have

$$A = a\,(r + 1)^t + a\,(r + 1)^{t-1} + a\,(r + 1)^{t-2} \ldots\ldots a\,(r + 1) \ldots\ldots$$
$$= a\,[(r + 1)^t + (r + 1)^{t-1} + (r + 1)^{t-2} \ldots\ldots (r + 1)]$$

But $(r + 1)^t$, $(r + 1)^{t-1}$, &c. is a geometrical progression, whose largest term is $(r + 1)^t$, the smallest $r + 1$, and the ratio $r + 1$. The sum of this progression, Art. XLVII. is

$$\frac{(r + 1)\;[(r + 1)^t - 1]}{r}.$$

$$\text{Hence } A = \frac{a\,(r + 1)\;[(r + 1)^t - 1]}{r}.$$

The same result may be obtained by another course of reasoning.

The amount of the sum a for one year is $a + a\,r$. Adding a to this, it becomes $2\,a + a\,r$.

The amount of this at the end of another year is $2\,a + a\,r + 2\,a\,r + a\,r^2$, or $2\,a + 3\,a\,r + a\,r^2$. Adding a to this it becomes

$$3\,a + 3\,a\,r + a\,r^2.$$

The amount of this for 1 year is

$$3\,a + 3\,a\,r + a\,r^2 + 3\,a\,r + 3\,a\,r^2 + a\,r^3,$$
$$= 3\,a + 6\,a\,r + 4\,a\,r^2 + a\,r^3,$$
$$= a\,(3 + 6\,r + 4\,r^2 + r^3).$$

This is the amount at the end of the third year before the addition is made to the capital. The law is now sufficiently manifest. With a little alteration, the quantity $3 + 6\,r + 4\,r^2 + r^3$ may be rendered the 4th power of $1 + r$. The three last coefficients are already right. If we add 1 to the quantity it becomes

$$4 + 6\,r + 4\,r^2 + r^3.$$

Multiply this by r and it becomes

$$4\,r + 6\,r^2 + 4\,r^3 + r^4.$$

Add 1 again and it becomes

$$1 + 4r + 6r^2 + 4r^3 + r^4.$$

This is now the 4th power of $1 + r$, and it may be written

$$(1 + r)^4.$$

Subtract the 1 which was added last, and it becomes

$$(1 + r)^4 - 1.$$

Divide this by r, because it was multiplied by r, and it becomes

$$\frac{(1 + r)^4 - 1}{r}.$$

Subtract 1 again, because 1 was added previous to multiplying by r; and it becomes

$$\frac{(1+r)^4 - 1}{r} - 1 = \frac{(1+r)^4 - (1+r)}{r} = \frac{(1+r)\,[(1+r)^3 - 1]}{r}$$

Substitute t in place of the exponent 3, and multiply by a, and it becomes

$$\frac{a\,(1+r)\,[(1+r)^t - 1]}{r} = A.$$

which is the same as before.

The particular question given above may now be solved by logarithms, using this formula.

log. $(1 + r) = 1.05$		0.021189
Multiply by $t = 15$		15
		105945
		.21189
log. $(1 + r)^{15} = 2.079$		.317835
Subtract 1	1	
log.	1.079	0.033021
log. $(1 + r)$		0.021189
log. $a = 10$		1.000000
Arith. Com. log. $r = .05$		1.301030
Ans. $226.59		2.355240

2. A man deposited annually $50 in a bank from the time his son was born, until he was 20 years of age; and it was taken out, together with compound interest on each deposit at 3 per cent., when his son was 21 years of age, and given to him. How much did the son receive?

3. How much did the bankers gain by receiving the money, supposing they were able to employ it all the time at 6 per cent. compound interest?

4. A man has a son 7 years old, and he wishes to give him $2000 when he is 21 years old; how much must he deposit annually at 4 per cent. compound interest, to be able to do it?

5. If a man deposits in a bank annually $35, in how long a time will it amount to $500 at 6 per cent. compound interest?

6. The first slaves were brought into the American Colonies in the year 1685. Suppose the first number to have been 50, and that 50 had been brought each year for 100 years, and the rate of increase 3 per cent. How many would there have been in the country at the end of the hundred years?

LIV. *Annuities.*

1. A man died leaving a legacy to a friend in the following manner; a sum of money was to be put at interest, such that, the person drawing 10 dollars a year, at the end of 15 years the principal and interest should both be exhausted. What sum must be put at interest at 6 per cent. to fulfil the above condition?

Let the learner generalize this example and form a rule; and then solve the following examples by it.

2. A man wishes to purchase an annuity which shall afford him $300 a year so long as he shall live. It is considered probable that he will live 30 years. What sum must he deposit in the annuity office to produce this sum, supposing he can be allowed 3 per cent. interest?

N. B. The principal and interest must be exhausted at the end of 30 years

3. If the man mentioned in the last example should die at the end of 18 years, how much would the annuity company gain?

4. If he were to live 43 years, how much would the company lose?

5. A man purchases an annuity for life, on the supposition that he shall live 45 years, for $15000, and is allowed 4 per cent. interest. How much must he draw annually that the whole may be exhausted?

6. A man has property to the amount of $35000, which yields him an income of 5 per cent. His annual expenses are $5000. How long will his property last him?

7. The number of slaves in the United States in 1810 was 1,191,000, and in 1820 the number was 1,531,000. What is the number at present, 1825, allowing the rate of increase to be the same?

8. There is a society established in the United States for the purpose of colonizing the free people of colour. Suppose the slaves to be emancipated as fast as this society can transport them away; how many must be sent away annually, that the number may be neither increased nor diminished?

9. How many must be sent away annually that the country may be cleared in 100 years?

10. If the colonization is not commenced till the year 1840, supposing the rate of increase to remain the same as from 1810 to 1820, how many must then be sent away annually, that the number remaining may continue the same?

11. How many must then be sent away annually, that the country may be cleared of them in 100 years?

Miscellaneous Examples.

1. An express set out to travel 240 miles in 4 days, but in consequence of the badness of the roads he found that he must go 5 miles the second day, 9 the third, and 14 the fourth, less than the first. How many miles must he travel each day?

2. Two workmen received the same sum for their labour; but if one had received 27 shillings more and the other 19 shillings less, then one would have received just three times as much as the other. What did they receive?

3. Two persons, A and B worked together, A worked 15 and B 18 days, and they received equal sums for their work. But if A had worked 17½ and B 14 days, then A would have received 35 shillings more than B. What was the daily wages of each?

4. Two merchants entered into a speculation, by which one gained 54 dollars more than the other. The whole gain was 49 dollars less than three times the gain of the less What were the gains?

5. A man bought a piece of cloth for a certain sum, and on measuring it, found that it cost him 8 dollars, but if there had been 4 yards more, it would have cost him only $7 per yard. How many yards were there?

6. Divide the number 46 into two such parts, that one of them being divided by 7, and the other by 3, the quotients may together be equal to 10.

7. A farm of 864 acres is divided between 3 persons. C has as many acres as A and B together; and the portions of A and B are in the proportion of 5 to 11. How many acres had each?

8. There are two numbers in the proportion of $\frac{1}{3}$ to $\frac{2}{5}$, the first of which being increased by 4 and the second by 6, they will be in the proportion of $\frac{2}{5}$ to $\frac{1}{2}$. What are the numbers?

9. A farmer has a stack of hay, from which he sells a quantity, which is to the quantity remaining in the proportion of 4 to 5. He then uses 15 loads, and finds that he has a quantity left, which is to the quantity sold as 1 to 2. How many loads did the stack at first contain?

10. There are 3 pieces of cloth, whose lengths are in the proportion of 3, 5, and 7; and 8 yards being cut off from each, the whole quantity is diminished in the proportion of 15 to 11. What was the length of each piece at first?

11. The number of days that 4 workmen were employed were severally as the numbers 4, 5, 6,7; their wages were the

same, viz. 3 shillings, and the sum received by the first and second was 36 shillings less than that received by the third and fourth. How much did each receive?

12. There are two numbers, the greater of which is three times the less; and the sum of their second powers is five times the sum of the numbers. What are the numbers?

13. What two numbers are those, of which the less is to the greater as 2 to 3; and whose product is six times the sum of the numbers?

14. There are two boys, the difference of whose ages is to their sum as 2 to 3, and their sum is to their product as 3 to 5. What are their ages?

15. A detachment of soldiers from a regiment being ordered to march on a particular service, each company furnished 4 times as many men as there were companies in the regiment; but these being found insufficient, each company furnished three more men, when their number was found to be increased in the proportion of 17 to 16. How many companies were there in the regiment?

16. Find two numbers which are in the proportion of 8 to 5, and whose product is 360.

17. A draper bought 2 pieces of cloth for $31.45, one being 50 and the other 65 cents per yard. He sold each at an advanced price of 12 cents per yard, and gained by the whole $6.36. What were the lengths of the pieces?

18. Two labourers, A and B, received $43.85 for their wages; A having been employed 15, and B 14 days; and A received for working four days $3.25 more than B for 3 days. What were their daily wages?

19. Having bought a certain quantity of brandy at 19 shillings per gallon, and a quantity of rum exceeding that of the brandy by 9 gallons, at 15 shillings per gallon, I find that I paid one shilling more for the brandy than for the rum. How many gallons were there of each?

20. Two persons, A and B, have each an annual income of $1200. A spends every year $120 more than B, and at the end of 4 years the amount of their savings is equal to one year's income of either. What does each spend annually?

21. In a naval engagement, the number of ships taken was 7 more, and the number burnt was 2 fewer, than the number sunk; 15 escaped, and the fleet consisted of 8 times the number sunk. Of how many did the fleet consist?

22. A cistern is filled in 50 minutes by 3 pipes, one of which conveys 10 gallons more, and the other 8 gallons less than the third, per minute. The cistern holds 1820 gallons. How much flows through each pipe per minute?

23. A farm of 750 acres is divided between three persons, A, B, and C. C has as much as A and B both, wanting 10 acres; and the shares of A and B are to each other in the proportion of 7 to 3. How many acres has each?

24. A certain sum of money being put at interest for 8 months, amounts to $772.50. The same sum put out at the same rate for 15 months amounts to 792.1875. Required the sum and the rate per cent.

25. From two casks of equal size are drawn quantities which are in the proportion of 5 to 8; and it appears that if 20 gallons less had been drawn from the one which now contains the less, only $\frac{4}{5}$ as much would have been drawn from it as from the other. How many gallons were drawn from each?

26. There are two pieces of land, which are in the form of rectangular parallelograms. The longer sides of the two are in the proportion of 6 to 11, and the adjacent sides of the less are in the proportion of 3 to 2. The whole distance round the less is 135 yards greater than the longer side of the larger piece. Required the sides of the less, and the longer side of the greater.

27. A person distributes forty shillings amongst fifty people, giving some 9d. and the rest 15d. each. How many were there of each?

28. Divide the number 49 into two such parts, that the quotient of the greater divided by the less, may be to the quotient of the less divided by the greater as $\frac{4}{3}$ to $\frac{3}{4}$.

29. A person put a certain sum to interest for 5 years, at 6 per cent. simple interest, and found that if he had put out the same sum for 8 years at $4\frac{1}{2}$ per cent. he would have received $60 more. What was the sum put out?

30. A regiment of militia containing 830 men is to be raised from three towns, A, B, and C. The contingents of A and B are in the proportion of 3 to 5; and of B and C in the proportion of 6 to 7. Required the numbers raised by each.

31. At what time between 6 and 7 o'clock are the hour and minute hands of a watch together?

32. There is a number consisting of two digits, the second of which is greater than the first; and if the number be divided by the sum of the digits, the quotient will be 4; but if the digits be inverted and that number divided by a number greater by 2 than the difference of the digits, the quotient will be 14. Required the number.

33. There is a fraction whose numerator being tripled, and the denominator diminished by 3, the value becomes $\frac{2}{3}$; but if the denominator be doubled and the numerator increased by 2, its value becomes $\frac{1}{7}$. Required the fraction.

34. A merchant bought a hogshead of wine for $100. A few gallons having leaked out, he sold the remainder for the original sum, thus gaining a sum per cent. on the cost of it, equal to twice the number of gallons which leaked out. How many gallons did he lose?

35. There are two pieces of cloth, differing in length 4 yards; the first is worth as many shillings per yard as the second contains yards; the second is worth as many shillings per yard as the first contains yards; and both pieces are worth £72. 10s. How many yards does each contain?

36. A merchant bought a piece of cloth for $180, and selling it at an advance of $1 a yard on the cost, he gained 15 per cent. Required the number of yards.

37. There are two rectangular pieces of land, whose lengths are to each other as 3 : 2, and surfaces as 5 : 3; the smaller one is 20 rods wide. What is the width of the other?

38. There is a cistern to be filled with a pump, by a man and a boy working at it alternately; the man would do it in 15 hours, the boy in 20. They filled it in 16 hours 48 minutes. How long did each work?

39. In a bag of money there is a certain number of eagles, as many quarter eagles, $\frac{3}{8}$ the number of half eagles, together

with dollars sufficient to make up the number of coins equal to $\frac{1}{5}$ of the value of the whole in dollars; and the number of eagles and dollars diminished by 2, is half the number of coins. What is the number of coins of each sort?

40. Suppose a man owes \$1000, what sum shall he pay daily so as to cancel the debt, principal and interest, at the end of a year, reckoning it at 6 per cent. simple interest?

41. A merchant bought two pieces of linen cloth, containing together 120 yards. He sold each piece for as many cents per yard as it contained yards, and found that one brought him in only $\frac{4}{9}$ as much as the other. How many yards were there in each piece?

42. A criminal having escaped from prison, travelled 10 hours before his escape was known. He was then pursued so as to be gained upon 3 miles an hour. After his pursuers had travelled 8 hours, they met an express going at the same rate as themselves, who met the criminal 2 hours and 24 min. before. In what time from the commencement of the pursuit will they overtake him?

43. A and B enter into partnership with a joint stock of \$900. A's capital was employed 4 months, and B's 7 months. When the stock and gain were divided, A received \$512, and B \$469. What was each man's stock?

44. A gentleman bought a rectangular lot of valuable land, giving 10 dollars for every foot in the perimeter. If the same quantity had been in a square, and he had bought it in the same way, it would have cost him \$33 less; and if he had bought a square piece of the same perimeter he would have had $12\frac{1}{4}$ rods more. What were the dimensions of the piece he bought?

45. A and B put to interest sums amounting together to 800 dollars. A's rate of interest was 1 per cent. more than B's, his yearly interest $\frac{5}{6}$ of B's; and at the end of 10 years his principal and simple interest amounted to $\frac{5}{7}$ of B's. What sum was put at interest by each, and at what rate?

46. Two messengers, A and B, were despatched at the same time to a place 90 miles distant; the former of whom riding one mile an hour more than the other, arrived at the end of his journey an hour before him. At what rate did each travel per hour?

47. A and B lay out some money on speculation. A disposes of his bargain for \$11, and gains as much per cent. as B

lays out; B's gain is \$36, and it appears that A gains four times as much per cent. as B. Required the capital of each?

48. A and B hired a pasture, into which A put four horses, and B as many as cost him 18 shillings a week. Afterwards B put in two additional horses, and found that he must pay 20 shillings a week. At what rate was the pasture hired?

49. A vintner draws a certain quantity of wine out of a full vessel that holds 256 gallons; and then filling the vessel with water, draws off the same quantity of liquor as before, and so on for four draughts, when there were only 81 gallons of pure wine left. How much wine did he draw each time?

50. Three merchants, A, B, and C, made a joint stock, by which they gained a sum less than that stock by \$80. A's share of the gain was \$60, his contribution to the stock was \$17 more than B's. B and C together contributed \$325. How much did each contribute?

51. A grocer sold 80lb. of mace and 100lb. of cloves for £65; but he sold 60 pounds more of cloves for £20 than he did of mace for £10. What was the price of a pound of each?

52. A and B, 165 miles distant from each other, set out with a design to meet. A travels one mile the first day, two the second, three the third, and so on; B travels 20 miles the first day, 18 the second, 16 the third, and so on. In how many days will they meet?

53. A and B engage to reap a field for \$20; and as A alone could reap it in 9 days, they promise to complete it in 5 days. They found however that they were obliged to call in C to assist them for the two last days, in consequence of which, B received $\frac{5}{6}$ of a dollar less than he otherwise would have done. In what time could B or C alone reap the field?

54. A mercer bought a piece of silk for \$54; and the number of shillings which he paid for a yard was $\frac{4}{9}$ of the number of yards. How many yards did he buy, and what was the price of a yard?

55. The fore wheel of a carriage makes 6 revolutions more than the hind wheel in going 120 yards; but if the periphery of each wheel be increased one yard, it will make only 4 revolutions more than the hind wheel in the same space. Required the circumference of each.

56. There are three numbers, the difference of whose differences is 5; the sum of the numbers is 20, and their continual product 130. Required the numbers.

57. From two towns, C and D, two travellers, A and B, set out to meet each other; and it appeared that when they met, B had gone 35 miles more than $\frac{3}{5}$ of the distance that A had travelled; but from their rate of travelling, A expected to reach C in $20\frac{5}{6}$ hours; and B to reach D in 30 hours. Required the distance from C to D.

58. Two men, A and B, entered into a speculation, to which B contributed $15 more than A. After four months, C was admitted, who added $50 to the stock; and at the end of 12 months from C's admission they had gained $159; when A withdrawing received for principal and gain $88. What did he originally subscribe?

59. The number of deaths in a besieged garrison amounted to 6 daily; and allowing for this diminution, their stock of provisions was sufficient to last 8 days. But on the evening of the sixth day, 100 men were killed in a sally, and afterwards the mortality increased to 10 daily. Supposing the stock of provisions unconsumed at the end of the sixth day to support 6 men for 61 days; it is required to find how long it would support the garrison, and the number of men alive when the provisions were exhausted.

60. There was a cask containing 20 gallons of brandy; a certain quantity of this was drawn off into another cask of equal size, and this last filled with water, and afterwards the first cask was filled with the mixture. It now appears that if $6\frac{2}{3}$ gallons of the mixture be drawn off from the first into the second cask, there will be equal quantities of brandy in each. Required the quantity of brandy first drawn off.

61. From two towns, C and D, which were at the distance of 396 miles, two persons, A and B, set out at the same time, and meet with each other, travelling as many days as are equal to the difference of the number of miles they travelled per day; when it appeared that A has travelled 216 miles. How many miles did each travel per day?

62. A tailor bought a piece of cloth for $200, from which he cut 5 yards for his own use, and sold the remainder for

$175, gaining 75 cents per yard. How many yards were there, and what did it cost him per yard?

63. There is a rectangular field containing 10 acres, 1 quarter, 5 rods, and the length of it exceeds the breadth by 12 rods. Required the dimensions of the field.

64. A man travelled 96 miles, and then found that if he had travelled 2 miles faster per hour, he should have been 8 hours less in performing the same journey. At what rate per hour did he travel?

65. A regiment of soldiers, consisting of 900 men, is formed into two squares, one of which has 6 men more in a side than the other. What is the number of men in a side of each square?

66. A and B travelled on the same road and at the same rate from Huntingdon to London. At the 50th mile stone from London, A overtook a drove of geese which were proceeding at the rate of three miles in two hours; and two hours afterwards met a stage waggon, which was moving at the rate of 9 miles in 4 hours. B overtook the same drove of geese at the 45th mile stone, and met the same stage waggon exactly forty minutes before he came to the 31st mile stone. Where was B when A reached London?

67. Two men, A and B, bought a farm consisting of 200 acres, for which they paid $200 each. On dividing the land, A says to B, if you will let me have my part in the situation which I shall choose, you shall have so much more land than I, that mine shall cost 75 cents per acre more than yours. B accepted the proposal. How much land did each have, and what was the price of each per acre?

68. A person bought two cubical stacks of hay for 41£; each of them cost as many shillings per solid yard as there were yards in a side of the other, and the greater stood on more ground than the less by 9 square yards. What was the price of each?

69. Two partners, A and B, dividing their gain $60 B took $20; A's money was in trade 4 months, and if the number 50 be divided by A's money, the quotient will give the number of months that B's money, which was $100, continued in trade. What was A's money, and how long did B's continue in trade?

END.

IMPROVED SCHOOL BOOKS.

Colburn's First Lessons, or, Intellectual Arithmetic.

THE merits of this little work are so well known, and so highly appreciated in Boston and its vicinity, that any recommendation of it is unnecessary, except to those parents and teachers in the country, to whom it has not been introduced. To such it may be interesting and important to be informed, that the system of which this work gives the elementary principles, is founded on this simple maxim; that, ***children should be instructed in every science, just so fast as they can understand it.*** In conformity with this principle, the book commences with examples so simple, that they can be perfectly comprehended and performed mentally by children of four or five years of age; having performed these, the scholar will be enabled to answer the more difficult questions which follow. He will find, at every stage of his progress, that what he has already done has perfectly prepared him for what is at present required. This will encourage him to proceed, and will afford him a satisfaction in his study, which can never be enjoyed while performing the merely mechanical operation of ciphering according to artificial rules.

This method entirely supersedes the necessity of any rules, and the book contains none. The scholar learns to reason correctly respecting all combinations of numbers; and if he reasons correctly, he must obtain the desired result. The scholar who can be made to understand how a sum *should* be done, needs neither book nor instructer to dictate how it *must* be done.

This admirable elementary Arithmetic introduces the scholar at once to that simple, practical system, which accords with the natural operations of the human mind. All that is learned in this way is precisely what will be found essential in transacting the ordinary business of life, and it prepares the way, in the best possible manner, for the more abstruse investigations which belong to maturer age. Children of five or six years of age will be able to make considerable progress in the science of numbers by pursuing this simple method of studying it, and it will uniformly be found that this is one of the most useful and interesting sciences upon which their minds can be occupied. By using this work children may be farther advanced at the age of nine or ten, than they can be at the age of fourteen or fifteen by the common method. Those who have used it, and are regarded as competent judges, have uniformly decided that more can be learned from it in one year, than can be acquired in two years from any other treatise ever published in America. Those who regard economy in time and money, cannot fail of holding a work in high estimation which will afford these important advantages.

Colburn's First Lessons are accompanied with such instructions as to the proper mode of using them, as will relieve parents and teachers from any embarrassment. The sale of the work has been so extensive that the publishers have been enabled so to reduce its price, that it is, at once, the cheapest and the best Arithmetic in the country.

Colburn's Sequel.

THIS work consists of two parts, in the first of which the author has given a great variety of questions, arranged according to the method pursued in the First Lessons; the second part consists of a few questions, with the solution of them, and such copious illustrations of the principles involved in the examples in the first part of the work, that the whole is rendered perfectly intelligible. The two parts are designed to be studied together. The answers to the questions in the first part are given in a Key, which is published separately for the use of instructers. If the scholar find any sum difficult, he must turn to the principles and illustrations, given in the second part, and these will furnish all the assistance that is needed.

The design of this arrangement is to make the scholar understand his subject thoroughly, instead of performing his sums by rule.

The First Lessons contain only examples of numbers so small, that they can be solved without the use of a slate. The Sequel commences with small and simple combinations, and proceeds gradually to the more extensive and varied, and the scholar will rarely have occasion for a principle in arithmetic which is not fully illustrated in this work.

Colburn's Introduction to Algebra.

THOSE who are competent to decide on the merits of this work consider it equal, at least, to either of the others composed by the same author.

The publishers cannot desire that it should have a higher commendation. The science of Algebra is so much simplified, that children may proceed with ease and advantage to the study of it, as soon as they have finished the preceding treatises on arithmetic. The same method is pursued in this as in the author's other works; every thing is made plain as he proceeds with his subject.

The uses which are performed by this science give it a high claim to more general attention. Few of the more abstract mathematical investigations can be conducted without it; and a great proportion of those, for which arithmetic is used, would be performed with much greater facility and accuracy by an algebraic process.

The study of Algebra is singularly adapted to discipline the mind, and give it direct and simple modes of reasoning, and it is universally regarded as one of the most pleasing studies in which the mind can be engaged.